Pricing equity-indexed annuities with path-dependent options

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Abstract

Equity-linked products such as equity-indexed annuities (EIAs) provide their customers with the greater of either the return linked to the underlying index or the minimum guaranteed return. The current volatile equity market increases the costs of options embedded in these products, and decreases the participation rates. This paper proposes four types of EIAs embedded with path-dependent options in order to increase participation rates. It also derives the joint distribution function of terminal time value and running maximum of Brownian motion. With the method of Esscher transforms, explicit pricing formulas for these proposed products and a floating-strike lookback option are derived.

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1. Introduction

Equity-linked products provide their customers with the greater of either the return linked to the underlying index or the minimum guaranteed return. Equity-linked products have recently gained popularity. This is in part due to economic changes in the financial market: relatively low interest rates and a bullish stock market which have led actuaries to design new types of saving products that link return to stock market performance. One of these products is equity-indexed annuities (EIAs). This paper is motivated by recent developments in EIAs sold in the United States. For detailed discussions on equity-linked products, see Streiff and DiBlase (1999), Tong (2000a), Kat (2001), Merrill and Thorley (1996), Moller (1998), and Lee (2001).

However, pricing equity-linked products is a challenging problem due to the complex payoff structure. Evaluating the guarantee embedded in these products is difficult and often requires advanced stochastic modeling techniques. Two key factors for pricing equity-linked products are the participation rate and the indexing method. The participation rate is the percentage of the index return to be credited. An indexing method is the method that the insurer or investment bank uses to calculate the index return to the customer based on the index values in the contract term.

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Tiong (2000b), using the method of Esscher transforms, has derived several closed-form formulas for pricing EIAs. As pointed out in the paper, the growth rates in sales have recently shown signs of slowing down because the current volatile equity market increases the costs of options embedded in EIAs and hence decreases the participation rates. Thus, new EIAs need to be designed that are similar to the existing ones but have a cheaper option and a higher participation rate. Hence, this paper introduces an up-and-in barrier EIA, an annual reset EIA with up-and-in barriers, a partial-time lookback EIA, and a partial lookback EIA with variable guarantee. In addition, this paper shall derive explicit pricing formulas for these proposed EIAs and a floating-strike lookback option by using the method of Esscher transforms.

2. Esscher transforms and some probability distributions

This section describes the method of Esscher transforms and derives the joint distribution function of terminal time value and partial-time running maximum of Brownian motion. Let \( S(t) \) denote the time- \( t \) price of an equity index. Assume that the index is constructed with all dividends reinvested. Assume that for \( t \geq 0 \),

\[
S(t) = S(0) e^{X(t)},
\]

where \( X(t) \) is a Brownian motion with drift \( \mu \) and diffusion coefficient \( \sigma \) and \( X(0) = 0 \). Thus the Brownian motion is a stochastic process with independent and stationary increments, and \( X(t) \) has a normal distribution with mean \( \mu t \) and variance \( \sigma^2 t \).

First, this section briefly summarizes a special case of the method of Esscher transforms developed by Gerber and Shiu (1994, 1996). For a nonzero real number \( h \), the moment generating function of \( X(t) \),

\[
E[e^{hX(t)}],
\]

exists for all \( t \geq 0 \), because \( \{X(t)\} \) is the Brownian motion as described above. The stochastic process

\[
\{e^{hX(t)}E[e^{hX(T)}]^{-1}\}
\]

is a positive martingale which can be used to define a new probability measure \( Q \). More precisely, this process is used to define the Radon–Nikodym derivative \( \frac{dQ}{dP} \), where \( P \) is the original probability measure. We call \( Q \) the Esscher measure of parameter \( h \).

For a random variable \( Y \) that is a real-valued function of \( \{X(t), 0 \leq t \leq T\} \), the expectation of \( Y \) under the new probability measure \( Q \) is calculated as

\[
E_Y \left[ \frac{e^{hX(T)}}{E[e^{hX(T)}]} \right] = E_Y \left[ \frac{1}{E[e^{hX(T)}]} \right] = \exp \left\{ (\mu + h\sigma^2)T + \frac{1}{2} \sigma^2 T^2 \right\},
\]

which will be denoted by \( E_Y \left[ e^{hX(T)}; h^* \right] \). The risk-neutral Esscher measure is the Esscher measure of parameter \( h^* = h^* \) under which the process \( \{e^{-rS(t)}\} \) is a martingale. Thus

\[
E_Y \left[ e^{hX(T)}; h^* \right] = S(0).
\]

Therefore, \( h^* \) is the solution of

\[
\mu + h^*\sigma^2 = r - \frac{1}{2}\sigma^2.
\]

For \( t \geq 0 \), the moment generating function of \( X(t) \) under the Esscher measure of parameter \( h \) is

\[
E[e^{hX(t)}; h] = \exp \left\{ (\mu + h\sigma^2)T + \frac{1}{2} \sigma^2 T^2 \right\},
\]

which implies that \( X(t) \) has a normal distribution with mean \( (\mu + h\sigma^2)t \) and variance \( \sigma^2 t \) under the Esscher measure. It can be shown that the process \( \{X(t)\} \) under the Esscher measure has independent and stationary increments. Thus, the process is a Brownian motion with drift \( \mu + h\sigma^2 \) and diffusion coefficient \( \sigma \) under the Esscher measure of parameter \( h \).
Now, let us consider a special case of the factorization formula (Gerber and Shiu, 1994, p 177; 1996, p. 188). For a random variable $Y$ that is a real-valued function of $\{X(t), 0 \leq t \leq T\}$,

$$E[e^{cX(T)}Y; h] = E[e^{cX(T)}; h]E[Y; h+c].$$

(2.5)

In particular, for an event $B$ whose condition is determined by $\{X(t), 0 \leq t \leq T\}$, the formula (2.5) can be expressed as follows:

$$E[e^{cX(T)I(B)}; h] = E[e^{cX(T)}; h]Pr(B; h+c),$$

(2.6)

where $I(\cdot)$ denotes the indicator function and $Pr(B; h)$ denotes the probability of the event $B$ under the Esscher measure of parameter $h$.

Now, let us discuss distributions and calculate some expectations to derive the joint distribution function of random variables $X(T)$ and $M(s,t)$. For $0 \leq s \leq t$, let

$$M(s,t) = \max\{X(\tau), s \leq \tau \leq t\}$$

(2.7)

be the maximum of the Brownian motion between time $s$ and time $t$. For simplicity, let $M(t) = M(0, t)$. It is known that for $t > 0$, $x \leq m$ and $m \geq 0$,

$$Pr(X(t) \leq x, M(t) \leq m) = \Phi \left( \frac{x - \mu t}{\sigma \sqrt{t}} \right) - e^{\frac{2\mu m}{\sigma^2}} \Phi \left( \frac{x - 2m - \mu t}{\sigma \sqrt{t}} \right),$$

(2.8)

where $\Phi(\cdot)$ denotes the standard normal distribution function. For a proof of (2.8), see Gerber and Shiu (2000). Obviously, for $x > m \geq 0$,

$$Pr(X(t) \leq x, M(t) \leq m) = Pr(M(t) \leq m) = \Phi \left( \frac{m - \mu t}{\sigma \sqrt{t}} \right) - e^{\frac{2\mu m}{\sigma^2}} \Phi \left( \frac{-m - \mu t}{\sigma \sqrt{t}} \right).$$

(2.9)

To generalize (2.8), consider a bivariate standard normal distribution. Note that

$$\Phi(a, b; \rho) = \Phi_2(b, a; \rho),$$

(2.10)

and

$$\Phi(a, b, \rho) = \Phi_2(a, b; -\rho),$$

(2.11)

where $\Phi_2(a, b; \rho)$ denotes the bivariate standard normal distribution function with correlation coefficient $\rho$. In Appendix A, we shall prove that, if $-\rho b + \sqrt{1-\rho^2} = a$ and $\rho \geq 0$, then

$$\Phi_2(a, b; -\rho) + \Phi_2(-a, c; \sqrt{1-\rho^2}) = \Phi(b)\Phi(c),$$

(2.12)

and

$$\Phi_2(a, b; -\rho) + \Phi(-b)\Phi(c) = \Phi_2(a, c; \sqrt{1-\rho^2}),$$

(2.13)

which will play a role in the proof of (2.22) and (5.14). Further applications of (2.12) and (2.13) have been considered by Lee (2002a).

Now, let $Z = (Z_1, Z_2, Z_3)$ have a standard trivariate normal distribution with correlation coefficients $\text{Corr}(Z_i, Z_j) = \rho_{ij}$ ($i, j = 1, 2, 3$). The distribution function of the random vector $Z$ is

$$\Phi(a, b, c; \rho_{12}, \rho_{13}, \rho_{23}) = Pr(Z_1 \leq a, Z_2 \leq b, Z_3 \leq c).$$

(2.14)
Note that
\[ \Phi_3(a, b, c; \rho_2, \rho_3, \rho_2) = \Phi_2(b, a, c; \rho_2, \rho_2) = \Phi_3(c, a, b; \rho_3, \rho_2, \rho_2) \]  
(2.15)

and
\[ \Phi_2(b, c; \rho_2) = \Phi_3(b, a, c; \rho_2, \rho_3, \rho_2) = \Phi_3(-a, b, c; -\rho_2, -\rho_3, \rho_2). \]  
(2.16)

Let us calculate the expectations necessary for deriving the joint distributions and pricing the proposed EIAs.

For numerical implementation of multivariate normal distribution functions, see Drezner (1978, 1994). Let random variable \( X \) be normal with mean \( \mu \) and variance \( \sigma^2 \). We assume that the random vector \((Z, W)\) is independent of \( X \). Then,
\[
E[I(X < a) \Phi_2\left( \frac{\theta X + c}{\sigma_1}, \frac{\theta X + c}{\sigma_2}, \rho \right)] = E[I(X < a) \Phi_2\left( \frac{\theta X + c}{\sigma_1}, \frac{\theta X + c}{\sigma_2}, \rho \right)] - E[I(X < a) \Phi_2(0, 0, 0)] = \Phi(\frac{a - \mu}{\sigma}) - \Phi(0),
\]
where \( \Phi \) denotes standard normal distribution function.

By the factorization formula (2.5), we obtain another expectation,
\[
E[\Phi(\frac{a - \mu}{\sigma}) - \Phi(0)] = \int_{-\infty}^{\infty} \Phi(\frac{a - \mu}{\sigma}) - \Phi(0) \, d\Phi_2(\alpha, \beta, \rho) = \frac{1}{2} \left( \Phi\left( \frac{a - \mu}{\sigma} \right) - \Phi(0) \right).
\]

In the particular case that \( c \) approaches infinity, it follows from (2.18) and (2.19) that
\[
E[\Phi(\frac{a - \mu}{\sigma}) - \Phi(0)] = e^{\ln(\sigma^2) / 2} \Phi_2\left( \frac{a - \mu}{\sigma}, \frac{b + \theta \mu_2}{k}, \theta \sigma_2^2, \rho \right).
\]  
(2.18)

where \( \mu \) denotes \( \mu + h\sigma^2 \).

Applying (2.18), we obtain another expectation,
\[
E[\Phi(\frac{a - \mu}{\sigma}) - \Phi(0)] = e^{\ln(\sigma^2) / 2} \Phi_2\left( \frac{a - \mu}{\sigma}, \frac{b + \theta \mu_2}{k}, \theta \sigma_2^2, \rho \right).
\]  
(2.19)
Furthermore, it is useful to know the following joint distribution function of random variables $M(t)$ and $X(T)$, $0 < t \leq T$.

\[
\begin{align*}
\Pr(M(t) \leq m, X(T) \leq x) &= \Phi_2 \left( \frac{x - \mu T}{\sigma \sqrt{T - t}}, \frac{m - \mu t}{\sigma \sqrt{t}} \right) - e^{\frac{2\mu}{\sigma^2}} m \Phi_2 \left( \frac{x - 2m - \mu T}{\sigma \sqrt{T}}, \frac{-m - \mu t}{\sigma \sqrt{t}} \right),
\end{align*}
\]

(2.22)

which generalizes (2.8). We shall give a proof of (2.22) in Appendix A.

Finally, we can derive a generalized version of (2.22). For $0 < s < t < T$, the joint distribution function of $X(T)$ and $M(s,t)$ can be calculated as follows:

\[
\begin{align*}
\Pr(M(s,t) \leq m, X(T) \leq x) &= E[\Pr(M(s,t) \leq m, X(T) \leq x | X(s))] \\
&= E[I(X(s) \leq m) \Pr(M(s,t) - X(s) \leq m - X(s), X(T) - X(s) \leq x - X(s)|X(s))].
\end{align*}
\]

(2.23)

3. Up-and-in barrier EIA

A point-to-point EIA measures the index increment from the start to the end of a contract term. The increment divided by the start index is the index return of the EIA. The payoff of the EIA is the greater of either the index return times a participation rate or a minimum guaranteed return. However, the payoff will be the same regardless of the path taken by the index to attain its final value. If customers believe that the underlying index will hit a barrier in a certain period, then they may not want to pay a high premium for the option embedded in a point-to-point EIA and will be reluctant to buy the EIA.

Consider barrier options, whose payoffs are the same as those of their underlying plain-vanilla options if the path of an underlying asset satisfies an activating condition, but will be zero otherwise. A barrier option is cheaper than its underlying plain-vanilla option because the payoff of the barrier option is less than or equal to that of the plain-vanilla option. Barrier options have been considered by Reiner and Rubinstein (1991), Heynen and Kat (1994b), Nelken (1996), Lin (1998), El Babbsiri and Noël (1998), Zhang (1998) and Kwok (1998). To increase the participation rate, let us now apply an up-and-in barrier option to new EIAs as an alternative to point-to-point EIAs. An up-and-in barrier EIA provides purchasers with the greater of the index return times the participation rate and
a minimum guaranteed return if the index rises above a barrier for the monitoring period and offers the minimum guaranteed return otherwise.

Let us take a close look at the payoff of the up-and-in barrier EIA. Assume that the minimum guaranteed return is $g$ for the contract term, the participation rate is $\alpha$, and the barrier is $B$. Let $u = \log(B/S(0))$ and $k = \log(1 + g/\alpha)$.

Then the payoff can be expressed as follows:

$$S(0)[(1 + a e^{X(T)} - 1)] - (1 + g),$$  \quad \text{if } X(T) > k, M(s, t) > u,

$$S(0)(1 + g),$$  \quad \text{if } X(T) \leq k, M(s, t) \leq u.

(3.1)

that is,

$$S(0)[(1 + \alpha e^{X(T)} - \alpha - g)I(X(T) > k, M(s, t) > u) + (\alpha + g) e^{-rT}P_1(\mu, T)].$$  \quad \text{(3.2)}

By the fundamental theorem of asset pricing, the time-0 value of the payoff (3.2) is

$$S(0)[(\alpha e^{X(T)} - \alpha - g)I(X(T) > k, M(s, t) > u) + (\alpha + g) e^{-rT}P_1(\mu, T)].$$  \quad \text{(3.3)}

4. Annual reset EIA with up-and-in barriers

The returns of annual reset EIAs are locked in periodically, such as annually, throughout the contract term. In each period, the annual reset EIAs provide customers with the greater of either the annual index return times the participation rate or a minimum guaranteed rate. Thus the annual reset EIAs credit better interest to the policy in a very volatile market. A drawback of annual reset EIAs is that the participation rate is relatively low because these EIAs have one embedded option in each policy year. Just as we previously applied up-and-in barrier options, let us now apply ones to annual reset EIAs to increase the participation rates.

Consider an annual reset EIA with up-and-in barriers. In each period, the EIA will provide customers with the greater of either the annual index return times the participation rate or a minimum guaranteed rate if the maximum index value for the monitoring period rises above a barrier. Otherwise, the EIA will credit to the policyholder the minimum guaranteed rate as annual return.
Let us take a close look at the total return of each period. Note that the total return is defined as the return plus one (Luenberger, 1998, p. 138). For \( 0 < s < t < T/n \), let 
\[
M_i = \max \{ X(\tau) - X(t_{i-1}), t_{i-1} + s \leq \tau \leq t_i - 1 + t \}, \quad i = 1, 2, \ldots, n,
\]
(4.1)
where \( t_i - t_{i-1} = T/n \), \( t_0 = 0 \) and \( t_n = T \). Assume that the minimum guaranteed rate is \( g \) for each period (from time \( t_{i-1} \) to time \( t_i \)), the participation rate is \( \alpha \) for each period, and \( k = \log(1 + g/\alpha) \).

For a real number \( u \), the level of the barrier in each period is \( S(t_{i-1})e^u \).

The total return of each period is as follows:
\[
1 + \alpha(e^{X_i} - 1), \quad \text{if } X_i > k \text{ and } M_i > u,
\]
\[
1 + g, \quad \text{otherwise}
\]
(4.2)
that is,
\[
(\alpha e^{X_i} - \alpha - g)I(X_i > k, M_i > u) + (1 + g),
\]
(4.3)
which has the same distribution as the total return of the up-and-in barrier EIA with maturity \( T/n \) and monitoring period from time \( s \) to time \( t \). Thus, the payoff of the annual reset EIA with up-and-in barriers is
\[
S(0) \prod_{i=1}^{n} [(\alpha e^{X_i} - \alpha - g)I(X_i > k, M_i > u) + (1 + g)].
\]
(4.4)

By the fundamental theorem of asset pricing, the time-0 value of the payoff (4.4) is
\[
S(0) e^{-rT}E\left[ \prod_{i=1}^{n} [(\alpha e^{X_i} - \alpha - g)I(X_i > k, M_i > u) + (1 + g)]; h^* \right].
\]
(4.5)

By the independence of \( \{(X_i, M_i), i = 1, 2, \ldots, n\} \), becomes
\[
S(0) \prod_{i=1}^{n} e^{-rT/n} E[(\alpha e^{X_i} - \alpha - g)I(X_i > k, M_i > u) + (1 + g); h^*] = S(0) \left[ V\left(1, \frac{T}{n}\right) \right]^*.
\]
(4.6)

5. Partial-time lookback EIA

A continuous lookback EIA credits to the policy interest based on the maximum index value attained during the life of the policy instead of the index value at maturity and offers a minimum guaranteed return if the maximum index value is low. Even in the case where the index has a high maximum value but drops substantially at maturity, the lookback EIA provides customers with a high return. However, the lookback option embedded in this EIA is expensive. For this reason, the participation rate of the lookback EIA is lower than that of the point-to-point EIA.

Consider a partial-time lookback EIA as an alternative to the lookback EIA. The index return of the partial-time lookback EIA is the same as that of the lookback EIA mentioned above except that the maximum index value is attained during the partial life of the policy. In other words, the index return is calculated as the difference between the maximum index value achieved during the final interval from time \( t \) to time \( T \) and the starting index value, divided by the starting index value.

Let us take a closer look at the payoff of the partial-time lookback EIA. Assume that the participation rate is \( \alpha \), the minimum guaranteed rate is \( g \), and \( k = \log(1 + g/\alpha) \). The payoff is as follows:
\[
S(0)(\alpha e^{M(t, T)} - 1) + 1), \quad \text{if } M(t, T) > k,
\]
\[
S(0)(g + 1), \quad \text{otherwise}
\]
(5.1)
Applying the fundamental theorem of asset pricing, the time-0 value of the payoff (5.2) is
\[ S(0) = e^{-rT}E[(e^{M(t,T)} - \alpha - g)I(M(t,T) > k) + (1 + g)]. \]  
(5.2)

Here, \( \Pr(M(t,T) > k) \) denotes the probability
\[ \Pr(M(t,T) > k) = 1 - \Pr(M(t,T) \leq k, X(T) \leq k) \]
\[ = 1 - \left[ \phi_2\left( \frac{-k - \mu T}{\sigma \sqrt{T}} - \frac{\xi}{\sigma \sqrt{T}} \right) - e^{\xi^2/2} \phi_2\left( \frac{-k - \mu T}{\sigma \sqrt{T}} + \frac{\xi}{\sigma \sqrt{T}} \right) \right]. \]  
(5.4)

Now, let us consider the second expectation of (5.3). It follows from (D21) and (D29) of Huang and Shiu (2001) that for a real number \( c, c + \xi \neq 0 \) and \( k \geq 0, \)
\[ E[e^{M(t,T)}I(M(t,T) > k)] = \frac{2e + \xi}{c + \xi} e^{\mu T + (1/2)\sigma^2 T} \Phi\left( \frac{-k + \mu T}{\sigma \sqrt{T}} + \frac{\xi}{\sigma \sqrt{T}} \right) + \frac{\xi}{c + \xi} e^{\xi^2/2} \Phi\left( \frac{-k + \mu T}{\sigma \sqrt{T}} - \frac{\xi}{\sigma \sqrt{T}} \right), \]  
(5.5)

where \( \xi \) denotes \( 2\mu \sigma^2. \) For the expectation on the right-hand side of (5.3), let us derive a generalized version of (5.5) as follows:
\[ E[e^{M(t,T)}I(M(t,T) > k)] = E[e^{M(t,T)}I(M(t,T) > k - X(t))|X(t)|], \]  
(5.6)

which can be divided into two iterated expectations,
\[ E[e^{M(t,T)}I(k \leq X(t)|E[e^{M(t,T) - X(t)}I(M(t,T) - X(t) > k - X(t))|X(t)]}], \]  
(5.7)

Because of the independence between random variables \( M(t,T) - X(t) \) and \( X(t), \) and the fact that the random variable \( M(T - t) \) has the same distribution as \( M(t,T) - X(t), \) the first conditional expectation of (5.7) is
\[ E[e^{M(t,T)}I(M(t,T) - X(t) > 0)] = E[e^{M(T - t)}I(M(T - t) > 0)], \]  
(5.8)

which becomes
\[ \frac{2e + \xi}{c + \xi} e^{\mu(T - t) + (1/2)\sigma^2 (T - t)} \Phi\left( \frac{-k + \mu (T - t)}{\sigma \sqrt{T - t}} + \frac{\xi}{\sigma \sqrt{T - t}} \right) + \frac{\xi}{c + \xi} e^{\xi^2/2} \Phi\left( \frac{-k + \mu (T - t)}{\sigma \sqrt{T - t}} - \frac{\xi}{\sigma \sqrt{T - t}} \right). \]  
(5.9)

According to (5.5) with \( t = T - t \) and \( k = 0. \) Thus the first expectation of (5.7) is
\[ E[e^{M(t,T)}I(k \leq X(t)|X(t))], \]  
(5.9)

\[ = \frac{2e + \xi}{c + \xi} e^{\mu(T - t) + (1/2)\sigma^2 (T - t)} \Phi\left( \frac{-k + \mu (T - t)}{\sigma \sqrt{T - t}} + \frac{\xi}{\sigma \sqrt{T - t}} \right) + \frac{\xi}{c + \xi} e^{\xi^2/2} \Phi\left( \frac{-k + \mu (T - t)}{\sigma \sqrt{T - t}} - \frac{\xi}{\sigma \sqrt{T - t}} \right). \]  
(5.10)
Now, consider the second expectation of (5.7). It follows from (5.5) with $k = k - X(t)$ and $t = T - t$ that the second conditional expectation of (5.7) is

$$
2c + \frac{e^{\mu(T-t)} + (1/2c)^2}{c + \xi} \Phi \left( \frac{X(t) - k + (\mu + c\sigma^2)(T-t)}{\sigma \sqrt{T-t}} \right) + \frac{\xi}{c + \xi} e^{\mu(t) + X(t)} \Phi \left( \frac{X(t) - k - \mu(T-t)}{\sigma \sqrt{T-t}} \right).
$$

(5.11)

Replacing the second conditional expectation of (5.7) with (5.11) and applying (2.20), we have the second expectation of (5.7),

$$
2c + \frac{e^{\mu(T-t)} + (1/2c)^2}{c + \xi} \Phi \left( k - (\mu + c\sigma^2)T \frac{k + \mu T}{\sigma \sqrt{T-t}} \right)
$$

(5.12)

Therefore, adding (5.12) to (5.10) and applying (2.13) to the sum of the first terms from (5.12) and (5.10), we obtain a generalized version of (5.5),

$$
E[e^{\mu T} I(M(T) > k)] = 2c + \frac{e^{\mu(T-t)} + (1/2c)^2}{c + \xi} \left[ \frac{k + \mu}{\sigma \sqrt{T-t}} \right] - \frac{k + \mu T}{\sigma \sqrt{T-t}} \Phi \left( \frac{\mu(T-t)}{\sigma \sqrt{T-t}} \right)
$$

(5.13)

6. Partial lookback EIA with variable guarantee

The previous three proposed EIAs reflect an attempt to offer the possibility of increased returns by increasing participation rates. This final alternative EIA proposes to increase the possible return in a different way: by introducing a variable guaranteed return in place of the fixed minimum guaranteed return.

Heynen and Kat (1994a) suggest a way of reducing the price of a lookback option while preserving some of its good qualities. The solution, they say, lies in a partial lookback option with a floating-strike that has a minimum value and that increases with the maximum value of an underlying asset. This section generalizes Heynen and Kat’s formula for this floating-strike lookback option and, applying the generalized option formula, derives a pricing formula for a partial lookback EIA with variable guarantee. For further discussions on lookback options, see Goldman et al. (1979), Conze and Viswanathan (1991), Bermin (2000).

Consider the payoff of the partial lookback EIA with variable guarantee. The variable guaranteed return of this EIA increases with the maximum index value for the monitoring period and is bounded below by the minimum guaranteed return. If the index return multiplied by the participation rate is bigger than the variable guaranteed return, it will be the return credited to the policy. Otherwise, the EIAs will provide customers with the variable

$$
S(0) e^{-rT} \left[ (r - \frac{1}{2} \sigma^2) I(1 - \mu + c\sigma^2) T \right] - (\alpha + \delta) P_2(\frac{\mu T}{\sigma \sqrt{T-t}}) + (1 + \gamma).
$$

(5.14)
guaranteed return. Thus, the partial lookback EIA with variable guarantee may be more attractive to customers who believe that the index will increase in a certain period because the EIA provides its policyholders with a higher guaranteed return.

Let us take a closer look at the payoff of the partial lookback EIA with variable guarantee. The payoff is as follows:

\[
S(t)(\alpha(\lambda e^{\max(X(t),0)}, L) - 1) + 1, \quad \text{if } e^{X(t)} > \lambda e^{\max(X(t),0)},
\]

\[
= S(t)(\alpha(\lambda e^{\max(X(t),0)}, L) - 1) + 1, \quad \text{otherwise},
\]

(6.1)

that is,

\[
S(0)(\alpha(\lambda e^{\max(M(I), L)} - 1) + 1)
\]

(6.2)

where \( g \) denotes the minimum guaranteed return for the contract period, \( I \) denotes \( \log\left(\frac{g}{\alpha + 1}\right) \), \( \lambda \) lies between zero and one, and 0 < \( s < T\). Note that the variable guaranteed return is \( g(\lambda e^{\max(M(s), L)} - 1) \).

To simplify writing, we define all expectations in this section as taken with respect to the risk-neutral Esscher measure. By the fundamental theorem of asset pricing, the time-0 value of the payoff (6.2) is

\[
S(0)(\alpha e^{-\mathcal{E}}[\lambda e^{\max(M(I), L)} - e^{X(T)}] + \alpha + e^{-\mathcal{T}}(1 - \alpha)),
\]

(6.3)

whose discounted expectation is a generalized formula for the floating-strike lookback put option (Heynen and Kat, 1997). It can be shown that the floating-strike lookback option price is

\[
e^{-\mathcal{E}}E[\lambda e^{\max(M(I), L)} - e^{X(T)}] = \Phi(f_1)\Phi(-f_2) + \Phi(f_1 - f_2)\Phi(-f_2) + \Phi(f_1 + f_2)\Phi(-f_2) - \Phi(f_1)
\]

(6.4)

where \( f_1 = \log\left(\frac{g}{\alpha} - 1\right) \) and \( f_2 = \log\left(\frac{g}{\alpha} + 1\right) \).

The time-0 price of the floating-strike lookback option with the monitoring period from time \( t \) to time \( (0 < t < T) \) can be expressed in the form of iterated expectations as follows:

\[
e^{-\mathcal{E}}E[\lambda e^{\max(M(t), L)} - e^{X(t)}] = e^{-\mathcal{E}}E[e^{X(t)}E[\lambda e^{\max(M(t), L)} - e^{X(t)}] | X(t)]
\]

(6.5)

which can be decomposed into the sum of two terms,

\[
e^{-\mathcal{E}}E[e^{X(t)} | X(t) \geq L]e^{-\mathcal{E}}E[\lambda e^{\max(M(t), L)} - e^{X(t)}] | X(t)]
\]

\[+ e^{-\mathcal{E}}E[e^{X(t)} | X(t) < L]e^{-\mathcal{E}}E[\lambda e^{\max(M(t), L)} - e^{X(t)}] | X(t)]
\]

(6.6)
First, let us consider the first term of (6.6). Applying the facts that the random vector \((M(s), t - X(s), X(T) - X(s))\) is independent of the random variable \(X(s)\) and that this random vector has the same distribution as the random vector \((M(s), X(T - s))\), the first term of (6.6) will be

\[
e^{-\theta t} E[e^{X(T)} I(X(s) \geq L)] e^{-\theta T} E[\xi \mathbb{P}(\xi = e^{X(T - s)})].
\]  

(6.7)

Applying the factorization formula (2.6), the first expectation of (6.7) is

\[
E[e^{X(T)} I(X(s) \geq L)] = E[e^{X(T)} I(X(s) \geq L; 1) = e^{\theta \xi} \mathbb{P}(\xi = 1),
\]  

(6.8)

where for \(i = 1, 2, g_i\) denotes \((-L + (r + (-1)^{t - 1}(1/2)s^2))\)/\(\sigma \sqrt{T - s}\). From (6.4) with \(L = 0, T = t - s\) and \(t = t - s\), the second discounted expectation of (6.7) is

\[
e^{-\theta r} E[\xi \mathbb{P}(\xi = e^{X(T - s)})] = -\Phi(k_i) \Phi \left(-e_1 + \frac{-\Phi(k_i)}{\sigma \sqrt{T - t}} \right) + \frac{-\lambda}{\sigma \sqrt{T - t}} \Phi \left(e_2 + \frac{\Phi(g_i)}{\sigma \sqrt{T - t}} \right)
\]  

(6.9)

where for \(i = 1, 2, h_i\) is \((r + (-1)^{t - 1}(1/2)s^2)\)(1 - \(s\))/\(\sigma \sqrt{T - s}\) and \(k_i\) is \((r + (-1)^{t - 1}(1/2)s^2)\)(1 - \(s\))/\(\sigma \sqrt{T - s}\).

Let us consider the second term of (6.6). It follows from the floating-strike lookback option formula (6.4) with \(L = L - X(s), T = T - s\) and \(t = t - s\) that the discounted conditional expectation in the second term of (6.6) is

\[
e^{-\theta r} E[\xi \mathbb{P}(\xi = e^{X(T) - X(s)}) \mathbb{P}(X(s) = X(T))] = -\Phi(K_i) \Phi \left(-e_1 + \frac{-\Phi(k_i)}{\sigma \sqrt{T - t}} \right) + \frac{-\lambda}{\sigma \sqrt{T - t}} \Phi \left(e_2 + \frac{\Phi(g_i)}{\sigma \sqrt{T - t}} \right)
\]  

(6.10)

where for \(i = 1, 2, K_i\) denotes \((X(s) - L + (r + (-1)^{t - 1}(1/2)s^2)(t - s))/\sigma \sqrt{T - s}\) and \(H_i\) is \((X(s) - L + (r + (-1)^{t - 1}(1/2)s^2)(t - s))/\sigma \sqrt{T - s}\). Thus, the second term of (6.6) is the discounted expectation of
\(e^{X(t)}I(X(t) < L)\) times (6.10). Applying (2.20) or (2.18) to this discounted expectation, the second term of (6.6) becomes

\[-\Phi\left(-g_1, f_1; -\sqrt{1 - \rho^2} \right) + \lambda e^{-rT} e^{2\sqrt{1 - \rho^2} \Phi_3 \left(d_1 + \frac{\log \lambda}{\sqrt{T}} - f_1, -g_1 \right)} - \Phi_3 \left(-d_1 + \frac{\log \lambda}{\sqrt{T}} - f_2, -g_2 \right).
\]

Therefore, the time-0 value of the floating-strike lookback option with monitoring period from time \(s\) to time \(t\) is

\[e^{-rT} E\left[\lambda e^{\max(M(s,t),L)} - e^X(T)\right] = (6.11) + \Phi(g_1)(6.9).
\]

If we place (6.12) into (6.3), we obtain the time-0 value of the partial lookback EIA with variable guarantee.

7. Conclusion

In this paper, to make EIAs more attractive in the deferred annuities market, we have proposed the up-and-in barrier EIA, the annual reset EIA with up-and-in barriers, the partial-time lookback EIA, and the partial lookback EIA with variable guarantee. These EIAs have higher participation rates than point-to-point, annual reset, and lookback EIAs, respectively. More realistic assumptions in pricing EIAs should be introduced in future research: stochastic interest rates, lapse rates, death benefits and transaction costs.

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Appendix A

Proof of (2.12) and (2.13). Let \(Z_1\) and \(Z_2\) follow the standard normal distribution independently. Then the random vectors \((-\rho Z_1 + \sqrt{1 - \rho^2} Z_2, Z_1)\) and \((\rho Z_1 - \sqrt{1 - \rho^2} Z_2, Z_2)\) will have the standard bivariate normal distributions with correlation coefficients \(-\rho\) and \(-\sqrt{1 - \rho^2}\), respectively. Thus the left-hand side of (2.12) is

\[
\Pr(-\rho Z_1 + \sqrt{1 - \rho^2} Z_2 \leq a, Z_1 \leq b) = \Pr(\rho Z_1 - \sqrt{1 - \rho^2} Z_2 \leq -a, Z_2 \leq c).
\]

(A.1)
The two events on the right-hand side of (A.1) are disjoint. Hence the two assumptions of (2.12) imply that the union of the two events becomes \( \{ Z_1 \leq b, Z_2 \leq c \} \). Thus, applying the independence of \( Z_1 \) and \( Z_2 \), the right-hand side of (A.1) is

\[
\Pr(Z_1 \leq b, Z_2 \leq c) = \Pr(Z_1 \leq b) \Pr(Z_2 \leq c) = \Phi(b)\Phi(c).
\]

Now, let us prove (2.13). Applying \( \Phi(-b) = 1 - \Phi(b) \), (2.12) and (2.11),

\[
\Phi_2(a, b; -\rho) + \Phi(-b)\Phi(c) = \Phi_2(a, b; -\rho) - \Phi(b)\Phi(c) + \Phi(c) = \Phi_2(a, c; -\sqrt{1 - \rho^2}).
\]

Proof of (2.22). Let \( f(y) \) denote the probability density function of the normal random variable with mean \( \mu(T-t) \) and variance \( \sigma^2(T-t) \). Applying the law of total probabilities, the joint probability distribution function of \( M(t) \) and \( X(T) \) is

\[
\Pr(M(t) \leq m, X(T) \leq x) = \int_{-\infty}^{\infty} \Pr(M(t) \leq m, X(T) \leq x \mid X(T) - X(t) = y)f(y) dy.
\]

which can be divided into the sum of two integrals,

\[
\int_{-\infty}^{x-m} \Pr(M(t) \leq m, X(T) \leq x \mid X(T) - X(t) = y)f(y) dy + \int_{x-m}^{\infty} \Pr(M(t) \leq m, X(T) \leq x \mid X(T) - X(t) = y)f(y) dy.
\]

Placing (2.9) into the first term of (A.4) and applying (2.8) and (2.21) to the second term of (A.4), the two integrals of (A.4) can be rewritten as

\[
\Phi_2(m - \mu\sigma\sqrt{T-t}, x - \mu(T-t)) - e^{2\rho/\sigma^2} \Phi_2(m - \mu\sigma\sqrt{T-t}, x - \mu(T-t) - \rho\sqrt{T-t})
\]

and

\[
\Phi_2\left(\frac{m - x + \mu(T-t)}{\sigma\sqrt{T-t}}, \frac{x - \mu(T-t)}{\sigma\sqrt{T-t}} - \sqrt{1 - \frac{\rho^2}{T}}\right) - e^{2\rho/\sigma^2} \Phi_2\left(\frac{m - x + \mu(T-t)}{\sigma\sqrt{T-t}}, \frac{x - \mu(T-t) - 2\mu T}{\sigma\sqrt{T-t}} - \sqrt{1 - \frac{\rho^2}{T}}\right),
\]

respectively. Hence, if we apply (2.13) to the sum of the first terms from (A.5) and (A.6) and apply (2.13) to the sum of the second terms from (A.5) and (A.6), then we have (2.22). □

References


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