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SOLVING A LARGE SCALE DISTRICTING PROBLEM: A CASE REPORT

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Scope and Purpose—The districting problem which consists in grouping basic geographical units into larger clusters, the "districts", occurs in various contexts: design of political districts, "Turfing" in telecommunications and design of sales territories. The last case is considered in this paper. The problem which arose in a German company for consumer goods is outlined and formulated as a mathematical programming model. A practicable procedure is developed based on the classical location-allocation approach where the optimal location of the centers of the districts and the allocation of the basic units to the districts alternate.

Abstract—The paper deals with the problem of defining the territories for 168 sales agents of a German manufacturer of consumer goods. About 1400 postal areas constitute the basic geographical units. The problem is solved by means of a location-allocation approach involving a standard code of a primal network algorithm as well as a new heuristic for resolving split areas. Numerical results and the implementation of the procedure as a planning tool are presented.

1. INTRODUCTION

The present paper deals with the definition of territories for a number of sales agents, a problem that arose in a major German company which manufactures consumer goods. The problem is presented in detail in the next section. Districting problems of this type have been a subject of the Operations Research literature for a long time, and various models and methods have been suggested. Section 3 briefly reviews the literature relevant to our problem. In Section 4 we propose a procedure for solving it, which is partly based on a classical location-allocation approach [1]. However, in view of the large size of the problem and the ill-conditioned data, we had to develop a new heuristic algorithm for eliminating the "split areas" resulting from the linear allocation model. This algorithm is outlined in Section 5. Section 6 reports on the numerical results and their practical use.

2. PROBLEM SETTING AND DATA

The problem concerns one division of the above mentioned company which delivers products to about 8000 wholesalers all over the Federal Republic of Germany. Sales promotion and advertising amongst the 70,000 retailers is very important in the considered business. This is the task of 168 agents who visit the retailers regularly, take care of product displays in the stores etc. but do not sell anything themselves. Nevertheless, for simplicity, we will refer to these agents as *salesmen* and to the retailers as *customers*.

Each salesman has a certain *territory*. The borders of the territories were fixed 8 years ago and are considered to be inappropriate for today's business, mainly because the distribution of workload is quite uneven. The workload caused by a single customer is expressed by an internal score taking into account the sales value and the frequency of visits. It is specified in the customer data file. The firm has postponed the necessary reorganization of territories several times, fearing the very expensive procedure which was required for the last reorganisation.

The firm was therefore looking for a new procedure for dividing the territory of the Federal

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Republic of Germany into a given number of sales territories, where the following aims – which could not be formulated more precisely by management – should be pursued:

1. The *size* of the territories, measured by the sum of the scores of the customers included in them, should be as uniform as possible. This aim can be expressed by upper and lower limits for the size of a territory, defined as the mean size plus or minus a *tolerance*, which is a strategic parameter. With the present structure, the tolerance is as large as 25%. This needs to be considerably reduced.
2. The territories may not overlap and should be of *compact shape*. This aim reflects the driving cost for visiting customers, which is difficult to express exactly, as well as the need for a clear organization. The term "compact" is not to be understood in a strict mathematical sense.
3. The procedure should be automatic, as far as possible. The effort required for subsequent manual adaptations of the results should be kept to a minimum. In particular, the definition of territories should allow for simple automatic allocation of the customers. Moreover, the procedure should be generally applicable to the other divisions, where the number of salesmen is quite different.

Due to the last requirement, we first decided to aggregate the customers into postal areas, which can be identified from the customer's address. The German postal code consists of four digits and, in addition for some towns, a two digit code for the town district. Area codes with the last digit unequal to zero refer to rural areas, some of which do not have a compact shape but surround other locations. Therefore, a further aggregation of these areas was carried out resulting in combinations of a small number of areas with consecutive codes. The smallest of the included codes is used as the code for the combined area. On the other hand, town districts are considered as separate areas for all towns with more than 100,000 inhabitants.

For the resulting areas, the scores of the customers could be easily aggregated from the existing customer files. The sum of the scores within an area is referred to as its *size*. As the absolute value of the scores is not of interest here, we will assume the mean size of the territories to be 100 and specify all other sizes as a percentage of this quantity in the following discussion. Table 1 summarizes the number and sizes of the areas.

The disadvantage of postal areas is that they are of very unequal size and not sufficiently fine in the urban regions. In particular, there are 25 cities of more than 100,000 habitants which have no postal districts. Thus, some of the areas have a size even larger than the mean territory size. It is expected that such cases will require a further manual subdivision of areas depending on the results of the automatic procedure. Also, the mean number of areas per territory, $1402/168 = 8.3$, is rather small, in comparison with similar studies in the literature (cf. Section 3). Refinements in the rural areas, however, are of little importance, as the table shows.

Finally, the present residences of the salesmen are available data, which can also be expressed by area code. However, management did not want these residences to influence the definition of the new territories heavily, because addresses can change frequently.

3. RELATED INVESTIGATIONS

The grouping of small geographic units or areas into larger geographic clusters according to some criteria is referred to in the literature as "districting". Many authors have investigated districting problems and provided models for applications in various contexts: design of political districts [2–4], "Turfing" in telecommunications [5], design of sales territories [6, 7 p. 175, 1, 8, 9] etc. The main purpose of most districting models is the design of a predetermined number of territories or districts with contiguous and compact shapes [3, p. 496] which are balanced

Table 1. Number and size of areas

	Rural areas	Urban areas	Town districts	All areas
Number of areas	734	428	240	1402
Mean size of an area	6.1	18.0	19.2	12.0
Maximal size of an area	25.6	100.2	112.8	112.8
Minimal size of an area	0.2	0.9	0.6	0.2

according to some activity measure and optimized using an objective function. Moreover, most formulations assume that an area may not be split between different territories (single source condition).

Political districting models use the area population as an activity measure to solve the "one man one vote" problem. In marketing applications, the sales potential and/or the anticipated workload in each area are used for the design of balanced sales territories. Similar to our case, the activity measure is usually predefined and calculated for each area. However, some authors [8, 10] integrate the ideas of territory design and sales resource allocation to maximize profit by applying a model hierarchy. First, a sales resource allocation model derives the profit maximizing workload for each area. Sales territories are then designed to balance these workloads. Zoltners and Sinha [9] point out that the employment of a single activity criterion may be a shortcoming for many applications and introduce a model for sales territories balanced according to multiple activity measures.

Spatial considerations are taken into account by minimizing a weighted distance between the areas and the territorial centers in order to design compact sales territories. Hess and Samuels [1] and Hess *et al.* [4] use squared Euclidean distances to enforce compactness. However, Cloonan [11] and Marlin [12] point out that travel costs in a territory are more proportional to simple straight line distances.

In [1, 4, 12] distances are calculated by using the coordinates of the area centers. In contrast, [5] and [9] represent the distances between the different areas by constructing an adjacency graph for every territorial center and by using a real road network. This approach can be very cumbersome for large-scale problems. Profit maximization formulations [8, 10, 13–16] usually do not treat spatial considerations directly but optimize the profitability of the territories. This is taken to be proportional to the total time spent by a salesman in each area. Distances can be expressed here as travel times from the salesmen's residences to the sales areas.

In general, the methods for solving districting problems employ a sequence of exact optimization routines and/or heuristics and exhibit the following pattern:

(a) Definition of one or several activity measures;
(b) Definition of n areas and calculation of the activity demand of each area;
(c) Selection of m points as territorial centers and the specification of their activity supply;
(d) Assignment of areas to territories in order to minimize (or maximize) some objective function.

The approaches to solving the assignment problem in step (d) can be divided into those that depend entirely upon heuristics and those that utilize more formalized mathematical programming techniques. Managerial heuristics have been provided in [6, 10]. Easingwood [6] recommends that sales management successively adjust the boundaries until workload is uniformly allocated. In Lodish's alignment procedure [10] management successively add and subtract areas from territories to reach equally distributed marginal profitability between the territories. In [2], a given territorial structure is improved stepwise by switching single areas between the territories.

Integer programming techniques have been applied in [3, 8, 9, 15]. In [3, 15], set-partitioning approaches are used to solve small districting problems. In a first step, feasible districts are constructed according to the activity constraints and, in a second step, districts are selected to optimize the objective function. Garfinkel and Nemhauser [3] present numerical results for problems with up to 55 population units. In [8, 9] general assignment formulations are applied. Zoltners [8] proposes the Ross and Soland approach as a possible way of solving his model without providing any numerical tests. Zoltners and Sinha [9] use a subgradient technique based on the Lagrangean Relaxation of the territory size constraints. Numerical results are reported for three examples with 13 territories and 204–280 areas. It should be emphasized that integer programming approaches ensure that every area is assigned to a unique territory but do not necessarily produce well balanced territories. In case of tight activity constraints, a feasible solution may not even exist (cf. Section 2). Another disadvantage of integer programming approaches is the computational effort required for large problems.

In [1, 4, 5, 12], linear transportation algorithms are used to solve the assignment problem. These algorithms, which are very suitable for large scale problems, yield optimal solutions which satisfy the activity constraints but can assign portions of the activity in one area to more than one territory. At most $m - 1$ split areas can arise in an optimal basic solution. These must then be resolved in a further step:

(e) Resolution of the split areas.

Marlin [12] points out that in particular cases splits may be retained, especially when the split area contains a large amount of activity. He suggests assigning the smaller areas without significantly violating the constraints and then solving the entire problem again. However, this procedure does not reduce the number of splits in general. Hess and Samuels [1] use a simple "tie breaking" heuristic which assigns an area to the territory with the maximum of the area's activity. In their applications they found, that a rate of $n/m \geqslant 20$ "is more than adequate to provide territories whose activity is within $\pm 10\%$ of average". Finally, Segal and Weinberger [5] offer an elaborate routine which uses total enumeration and considers three different objectives, the minimization of either the maximum absolute deviation, the second worst deviation or the sum of the absolute deviation from the average territorial activity. They point out that their procedure works well "if the number of splits is less than 15 or so". However, they do not provide any numerical results.

Most of the procedures assume fixed territorial centers (e.g. residences of the salesmen). In contrast [1] and [4] permit variable center locations in order to optimize the total territorial structure. The center locations are determined by improving the initial values in the following steps:

(f) Redefinition of the territorial centers according to the current assignment;
(g) Repeat of steps (d)–(f) until no significant improvement is noted.

A main advantage of squared Euclidean distances [1, 4] is that in step (f) optimal centers can be very easily calculated as the centers of gravity of every territory. Hess and Samuels [4] report on many successful real life applications of this location–allocation approach known as "GEOLINE".

4. MODEL FORMULATION AND SOLUTION PROCEDURE

According to the management requirements as outlined in Section 2, we formulate the problem as a planar transportation–location model [17]. The condition that every area is to be assigned to a unique territory is considered by an additional *single source constraint*. To enforce compactness of the territories we use weighted squared Euclidean distances as a minimization criterion, similar to the GEOLINE procedure of Hess and Samuels [1]. To consider the geographical interrelation of the large number of areas and to calculate the distances, we use *geographical coordinates*, which, in the case of rural areas, refer to the center or to the main location.

We solve the problem by applying a location–allocation procedure. The "single source" transportation problem in the allocation part could be solved exactly by an algorithm similar to that of Nagelhout and Thompson [18]. However, the dimensions of our problem and the data structure, which does not ensure a feasible solution in the single source case (cf. Section 2), motivated us to apply a linear transportation algorithm and to resolve the splits in a final step.

Using the following notation:

Data
- m: number of territories
- n: number of areas
- I: set of territories
- J: set of areas
- n_j: north coordinate of area j
- e_j: east coordinate of area j
- b_j: size (scores) of area j
- $\bar{A} = \sum_j b_j/m$: mean size of territory
- t: tolerance for the territory size
- $A_{\min} = \bar{A} - t$; $A_{\max} = \bar{A} + t$: lower and upper limit for the territory size

Variables
- x_{ij}: amount of scores in area j assigned to territory i
- N_i: north coordinate of the center of territory i
- E_i: east coordinate of the center of territory i

we formulate the model as follows:

minimize
$$\sum_{i \in I} \sum_{j \in J} x_{ij}[(N_i - n_j)^2 + (E_i - e_j)^2] \tag{4.1}$$

subject to

$$A_{\min} \leqslant \sum_{j \in J} x_{ij} \leqslant A_{\max} \qquad (i \in I) \tag{4.2}$$

$$\sum_{i \in I} x_{ij} = b_j \qquad (j \in J) \tag{4.3}$$

$$x_{ij} \in \{0, b_j\} \qquad (i \in I, j \in J). \tag{4.4}$$

For given center locations $(\bar{N}_i, \bar{E}_i)$ and relaxed conditions (4.4) the above model reduces to the transportation problem

$$\left.\begin{array}{ll} \text{minimize} & \sum_{i \in I} \sum_{j \in J} x_{ij} d_{ij} \\ \text{subject to (4.2), (4.3)} & \\ \text{and} & x_{ij} \geqslant 0 \end{array}\right\} \tag{4.5}$$

where

$$d_{ij} = [(\bar{N}_i - n_j)^2 + (\bar{E}_i - e_j)^2] \tag{4.6}$$

To solve (4.1)–(4.4) we applied the following location–allocation procedure. The superscript k denotes the iteration counter, $k_{\max}$ the maximal number of iterations.

Step 1: Initialize; set $k_{\max}$; $k = 1$; set $(\bar{N}_i^1, \bar{E}_i^1)$ equal to the present salesmen residences.
Step 2: Calculate the distances d_{ij}^k according to (4.6) and solve (4.5). The optimal solution is (x_{ij}^k).
Step 3: Determine the new coordinates for the centers of the territories by

$$\bar{N}_i^{k+1} = \frac{\sum_j x_{ij}^k n_j}{\sum_j x_{ij}^k}, \qquad \bar{E}_i^{k+1} = \frac{\sum_j x_{ij}^k e_j}{\sum_j x_{ij}^k}$$

Step 4: *Test*: If the territory centers have changed and $k < k_{\max}$, set $k = k + 1$ and go to step 2; otherwise, go to step 5.
Step 5: Resolve the splits of the last solution (x_{ij}^k); stop.

Problem (4.5) in step 2 was solved by the primal network code NET [19], which is suitable for large capacitated transportation problems. For this purpose the following network formulation of (4.5) was used (Fig. 1):

—one source node for every territory with supply $A_{\max}$
—one destination node for every area j with demand b_j

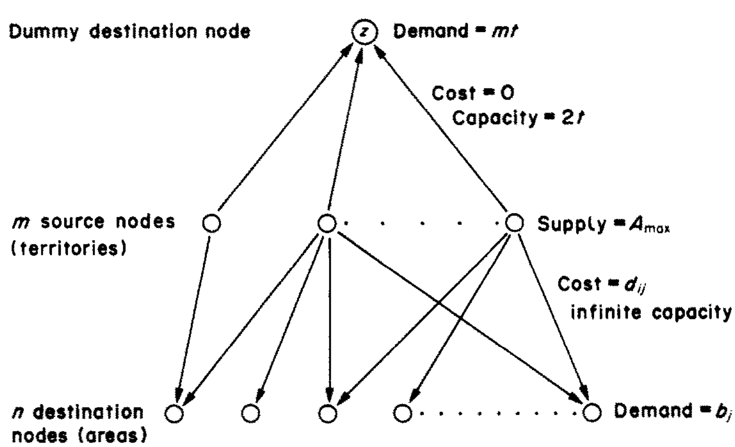

Fig. 1. Network formulation of the capacitated transportation problem (4.5).

—one dummy destination node (z) for the surplus supply with demand $m \cdot t$
—one arc from every source node i to every destination node j, if $\sqrt{d_{ij}} \leqslant d_{\max}$, with cost d_{ij} and infinite capacity
—one arc from every source node to z with cost 0 and capacity $2t$.

In order to reduce the number of arcs, different values of the maximal feasible distance $d_{\max}$ were tested. Finally, 80 kilometers were chosen as an appropriate measure, which yields approximately 20,000 arcs. Procedures for the solution of split areas in step 5 are described in the next section.

5. RESOLVING THE SPLITS

5.1. *Problem formulation*

The procedure described so far yields an optimal basic solution $(\bar{x}_{ij})$ of (4.5) with at most $m-1$ split areas j, i.e. such that $\bar{x}_{ij} > 0$ for more than one territory i. When resolving the splits, we will preserve the main goal reached in the previous steps, the compact shape of the territories, by the following condition: a split area may only be assigned to a territory i if $\bar{x}_{ij} > 0$. In this case we call i and j *adjacent*. Moreover, the territory sizes are restricted again by (4.2). The objective is now, according to the original aim no. 3 (cf. Section 2) to resolve as many splits as possible, thus minimizing the number of manual corrections. A perfect resolution of all splits is not possible under the above constraints. With the notation

J': set of split areas,
I': set of territories adjacent to any split area,
F: the non-oriented graph with the vertex set $I' \cup J'$ and the above adjacency (a forest, since $(\bar{x}_{ij})$ is a basic solution),
U_v: set of vertices adjacent to vertex v of F,
$U(V)$ set of vertices adjacent to any vertex $v \in V$,
a_i: $= \sum_{j \in J \setminus J'} \bar{x}_{ij}$ size of territory i without split areas;

we have to solve the problem:

Maximize the number of edges (i, j) such that $x_{ij} = b_j$, subject to

$$\left.\begin{aligned} &\sum_{i \in U_j} x_{ij} = b_j \qquad (j \in J') \\ A_{\min} \leqslant a_i + &\sum_{j \in U_i} x_{ij} \leqslant A_{\max} \qquad (i \in I') \\ &x_{ij} \geqslant 0 \qquad (i \in I', j \in J'). \end{aligned}\right\} \tag{5.1}$$

Note that $(\bar{x}_{ij})$ is a feasible solution of (5.1).

The simple tie breaking heuristic [1] (cf. Section 3), which was first tried, gave very poor results: for about 50% of the territories the size restriction (5.1) was violated, in many cases heavily. We then used a heuristic involving the transportation problem (4.5) outlined in Section 5.2, but too many splits remained unresolved (cf. Section 6). Finally, we developed a new heuristic using the tree structure of F for performing single feasible assignments. This will be presented in Section 5.3.

5.2. *A heuristic based on the transportation problem*

This heuristic simply consists of solving the transportation problem (4.5) repeatedly with the final centre locations fixed, and increasing the tolerance t stepwise by an increment Δ, starting from $t = 0$ up to a given limit. Each iteration includes the steps:

(a) Cancel all edges in the network of Fig. 1, which are non-basic in the current solution, except those ending in the dummy destination. These edges correspond to non-adjacent territories and areas.
(b) Set $A_{\min} := A_{\min} - \Delta$, $A_{\max} := A_{\max} + \Delta$. Thus, the supply in each source increases by Δ, the demand in z by $m\Delta$ and the capacity of each edge (i, z) by 2Δ. Hence the current solution is no longer feasible nor basic since the saturated edges (i, z) lose this property.
(c) Solve (4.5) again.

As the number B of basic edges (i, j) $(j \neq z)$ can only decrease in step (c), the number of splits which equals $B - n$, can only decrease.

5.3. *A local assignment heuristic*

We increase single variables x_{ij} $(i \in I', j \in J')$ stepwise, starting from $x_{ij} = 0$, in either of two ways:

(a) An obligatory *partial assignment* takes place, if for some (i, j)

where

$$\left.\begin{aligned} \delta_{ij} &= \max(\delta_1, \delta_2) > 0, \\ \delta_1 &= b_j - \sum_{h \in U_j, h \neq i} (A_{\max} - a_h) \\ \delta_2 &= A_{\min} - a_i - \sum_{k \in U_i, k \neq j} b_k \end{aligned}\right\} \tag{5.2}$$

because (5.1) implies $x_{ij} \geq \delta_{ij}$ for any feasible solution.

(b) An arbitrary *full assignment* of area j_0 to territory i_0 takes place, if it proves to be feasible, i.e. leads to a feasible solution or to a solution which can be made feasible by further increases of variables. An obvious sufficient condition for this is

and

$$\begin{aligned} a_{i_0} + \sum_{j \in U_{i_0}} b_j &\leq A_{\max} \\ a_i \geq A_{\min} \qquad &(i \in U_{j_0}, i \neq i_0). \end{aligned} \tag{5.3}$$

However, this condition is very strong and generally allows only a few assignments. The following theorem states a sufficient and necessary condition, assuming that all necessary partial assignments have been made.

Theorem

If (5.1) has a solution and $\delta_{ij} \leq 0$ for any $i \in I'$, $j \in J'$, then there exists a solution (x_{ij}) of (5.1) such that $x_{i_0 j_0} = b_{j_0}$ if and only if

$$a_{i_0} + b_{j_0} \leq A_{\max} \tag{5.4}$$

Proof: The necessity of (5.4) is obvious. From the theorem of Gale on the feasibility of the network model in Fig. 1, it follows that (5.1) is feasible if and only if

$$\sum_{j \in J_1} b_j \leq \sum_{i \in U(J_1)} (A_{\max} - a_i) \tag{5.5}$$

for any $J_1 \subset J'$ such that $J_1 \cup U(J_1)$ is connected in F, and

$$\sum_{j \in U(I_1)} b_j \geq \sum_{i \in I_1} (A_{\min} - a_i) \tag{5.6}$$

for any $I_1 \subset I'$ such that $I_1 \cup U(I_1)$ is connected in F, because (5.5) expresses that the demand does not exceed the supply in the set $J_1 \cup U(J_1)$ with no entering edges, and (5.6) expresses that the supply minus the capacity of the outgoing edges (i, z) does not exceed the demand in $I_1 \cup U(I_1)$, all other connected sets of vertices having entering and outgoing edges of infinite capacity.

The additional condition $x_{i_0 j_0} = b_{j_0}$ reduces both the supply in i_0 and the demand in j_0 by b_{j_0}. Hence it preserves the feasibility if there is a slack of at least b_{j_0} in (5.5) for any J_1 such that $j_0 \notin J_1$ and $i_0 \in U(J_1)$, and in (5.6) for any I_1 such that $i_0 \notin I_1$ and $j_0 \in U(I_1)$. Let I_1, J_1 be such sets and q_v the immediate successor of a vertex $v \in I_1 \cup J_1$ on the unique path in F from v to i_0. Then, (cf. Fig. 2) the right hand side of (5.5) is equal to

$$\sum_{j \in J_1} \sum_{\substack{i \in U_j \\ i \neq q_j}} (A_{\max} - a_i) + A_{\max} - a_{i_0} \geq \sum_{j \in J_1} b_j + b_{j_0}$$

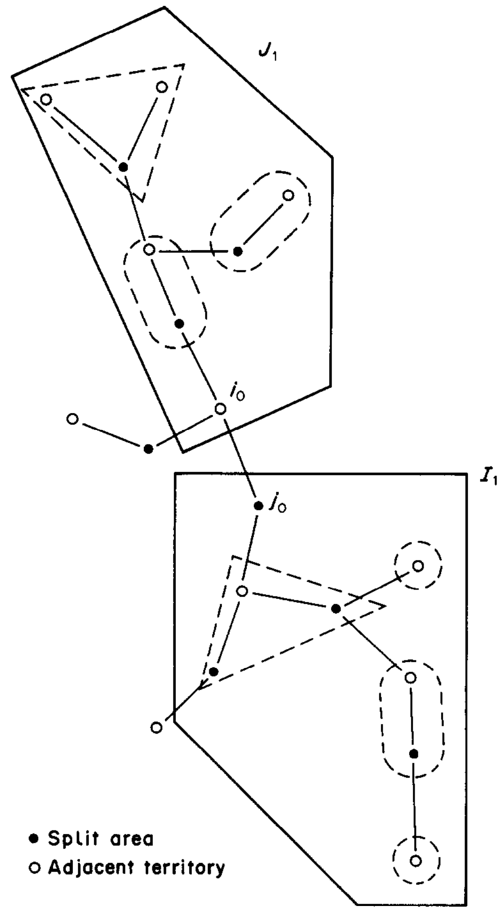

Fig. 2. Illustration of the proof. The dotted lines surround the vertex sets $\{i \in U_j : i \neq q_j\}$ for $j \in J_1$ and $\{j \in U_i : j \neq q_i\}$ for $i \in I_1$.

as $\delta_{q_j j} \leqslant 0$ and (5.4), and the left hand side of (5.6) is equal to

$$\sum_{i \in I_1} \sum_{\substack{j \in U_i \\ j \neq q_i}} b_j + b_{j_0} \geqslant \sum_{i \in I_1} (A_{\min} - a_i) + b_{j_0},$$

as $\delta_{iq_i} \leqslant 0$ and (5.4).

The idea of the procedure is first to perform partial assignments for any adjacent i, j with $\delta_{ij} > 0$, and as soon as all $\delta_{ij} \leqslant 0$, full assignments using the theorem. Each assignment increases the size a_i of the territory i and reduces the quantity

$\tilde{b}_j$: scores of area j, which are not yet assigned.

Replacing b_j by $\tilde{b}_j$ in (5.4), the theorem can be applied recursively to any (i_0, j_0) such that

$$\tilde{b}_{j_0} > 0 \qquad \text{and} \qquad x_{i_0 j_0} + \tilde{b}_{j_0} = b_{j_0} \tag{5.7}$$

i.e. there has not yet been an assignment of area j_0 to any $i \neq i_0$. Using the variables

$$\alpha_i = a_i + \sum_{j \in U_i} \tilde{b}_j \qquad (i \in I')$$

$$\beta_j = \sum_{i \in U_j} (A_{\max} - a_i) - \tilde{b}_j \qquad (j \in J')$$

the calculation of δ_{ij} is simplified as follows:

$$\delta_{ij} = \max(\alpha_i - \tilde{b}_j - A_{\min}, \beta_j + a_i - A_{\max}).$$

Then, each partial or full assignment between j and i includes the operations, with $\delta = \delta_{ij}$ *or* $\delta = \tilde{b}_{j_0}$, respectively

$$\left.\begin{array}{lll} x_{ij} := x_{ij} + \delta, & a_i := a_i + \delta, & \tilde{b}_j := \tilde{b}_j - \delta, \\ \alpha_h := \alpha_h - \delta \quad (h \in U_j, h \neq i), & & \beta_k := \beta_k - \delta \quad (k \in U_i, k \neq j). \end{array}\right\} \tag{5.8}$$

Unfortunately, these operations may again violate the condition $\delta_{hk} \leqslant 0$ for $h \in U_i$ or $k \in U_j$. Hence it is necessary to check δ_{ij} for all edges (i, j) of F not only once, but in repeated runs. A full assignment according to the theorem is only possible after a complete run through all edges where no assignment has occurred.

The efficiency of the procedure can be improved by the following organisation:

1. Introduce an orientation of F such that each component is a rooted tree, and introduce a topological ordering of the vertices, excluding the roots, i.e. such that a vertex v is positioned before its unique successor q_v, and process the edges (v, q_v) in each run according to this order.
2. At each assignment (5.8), label the immediate predecessors h of j and k of i. At the second and all further runs, check only edges (v, q_v) where q_v is labeled; for it is easy to see that the violation of $\delta_{ij} \leqslant 0$ cannot propagate downstream again.
3. Even if $\delta_{ij} \leqslant 0$ has not yet been established for all i, j, the condition (5.3) for a full assignment can be weakened, observing that the scores $\tilde{b}_{j_0}$ are disposable for any assignment in the current run, as soon as $i = q_{j_0}$ has been reached. Then, the full assignment of j_0 to i_0 is feasible, if (5.7) holds and if

$$\left.\begin{array}{llll} \tilde{b}_{j_0} + a_i + \tilde{b}_{q_i} \leqslant A_{\max}, & \text{for} & i_0 = i & \\ \tilde{b}_{j_0} + a_{i_0} \leqslant A_{\max} & \text{and} & a_i \geqslant A_{\min}, & \text{for} \quad i_0 \text{ proceeding } j_0. \end{array}\right\} \tag{5.9}$$

Thus, the procedure consists of three phases:

Phase 1: A first run through all edges with partial assignments and full assignments according to (5.9).

Phase 2: Further runs with only partial assignments according to point 2 above, until $\delta_{ij} \leqslant 0$ for all i, j.

Phase 3: Check the edges for a full assignment according to the theorem. If none is possible, the procedure stops. After each assignment return to phase 2.

This procedure can be implemented with a complexity of $O(m^2)$ as the following arguments show: let M be the number of edges of F, then $M \leqslant 2m - 2$. In phase 1, M edges are checked and at most M assignments (5.8) occur, each with at most M operations. In phase 3, at most M assignments occur, each requiring at most M preceeding checks. Every assignment in phase 1 or 3 entails at most one operation for every upstream edge in phase 2, either labeling its initial vertex in (5.8) or checking it, when the terminal vertex is labeled.

6. RESULTS

The solution procedure was programmed in FORTRAN 77 and tested on a SIEMENS 7882 mainframe computer (12 MIPS) at the University of Hamburg.

The time for the solution of the transportation problems varies between 10 s for the first iteration and approximately 6 s for all other iterations of the location–allocation procedure (cf. Section 4). The split resolving heuristic described in Section 5.3 needs only 1 s CPU-time.

In order to verify compactness and to facilitate the presentation of the results the solutions were plotted in a frame of 140 cm × 85 cm, where each territory is represented by a star with a straight line between the center and each adjacent area, which can be identified by its postal code (cf. Fig. 3). Regions with a high area density were additionally plotted on a larger scale.

A first run with fixed territorial centers according to the present salesmen residences, showed the inappropriate geographical distribution of the salesmen in comparison with workload in the present structure. The plotted figures resembled comets with their tails in the south rather than stars (cf. Fig. 4), due to the large number of salesmen in the north and the large amount of scores in the south. This structure changes essentially after the first redefinition of the territorial centers. The solution of the second transportation problem already shows a good star structure. The

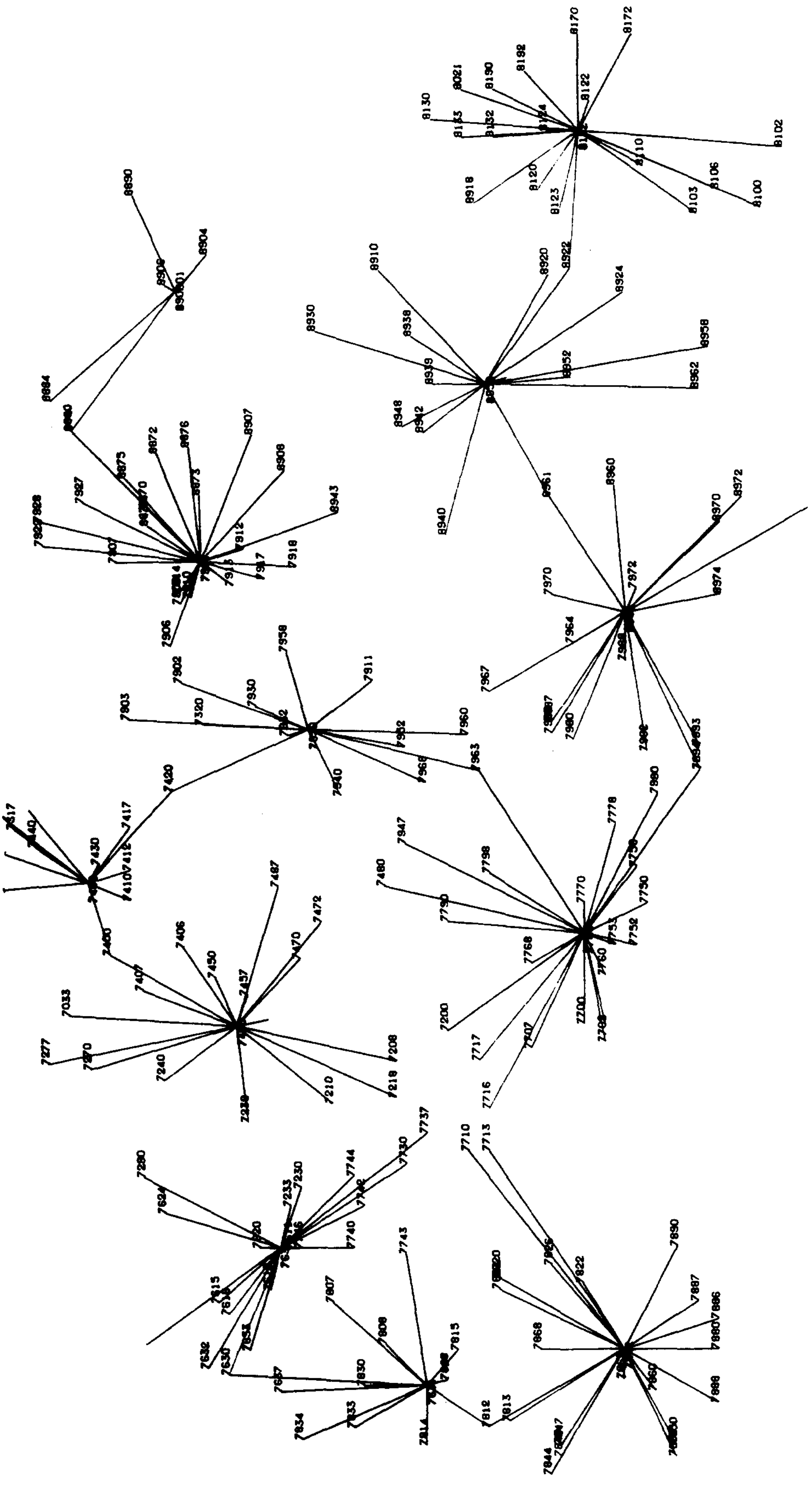

Fig. 3. Some plotted territories (before resolving the splits).

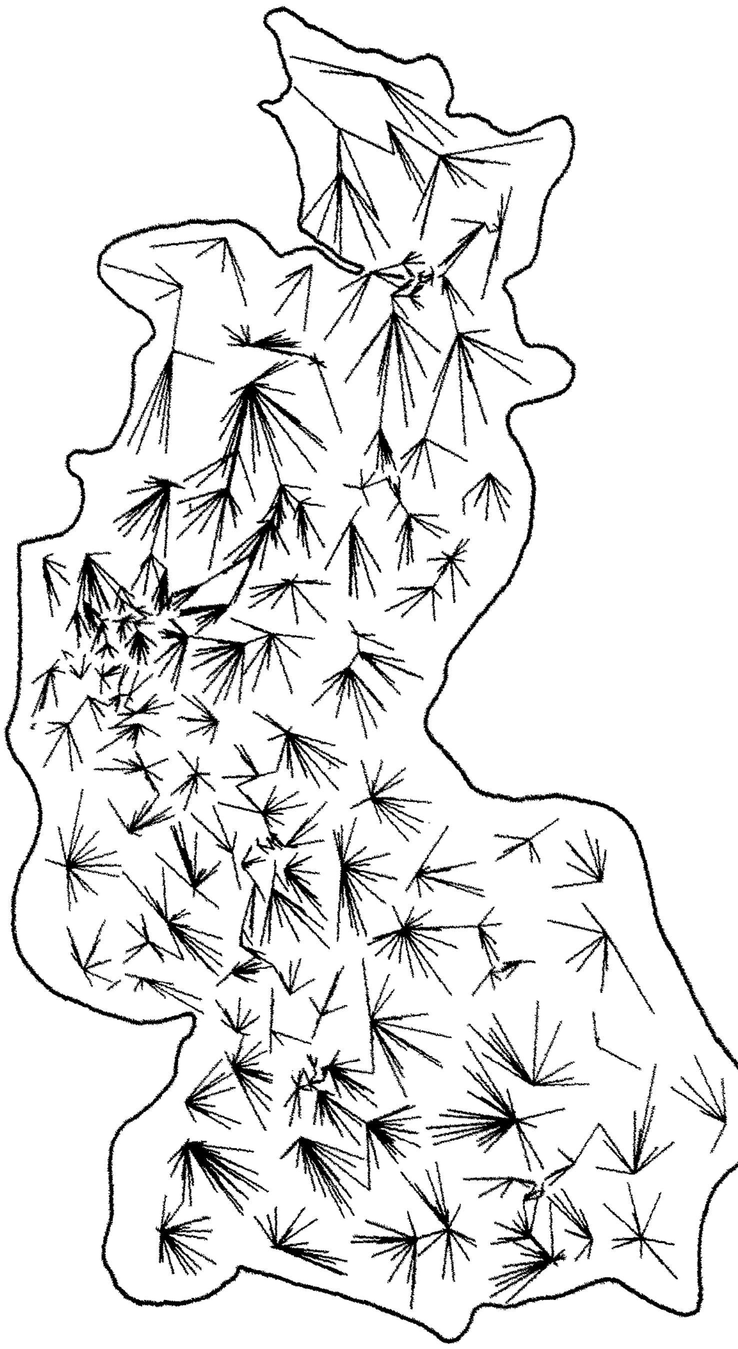

Fig. 4. Territories with fixed centers (splits resolved).

improvements in the additional location–allocation steps are rather marginal, so we decided to terminate the procedure after at most 9 iterations even without reaching a stationary solution. Figures 4 and 5 show the general structure of the territories after resolving the splits, with fixed centers and after 9 iterations, respectively. An entire run needs approximately 270 s CPU-time, the main part of which is required by network generation taking 20 s in each allocation step. However, the number of iterations could be reduced to up to 4 without significant changes in the solution structure, when CPU-time is a problem.

The results of both of the split resolving heuristics are affected by the parameters of the previous allocation procedure, i.e. the limits A_{min}, A_{max} in (4.5) and the number k_{max} of location iterations: the best results are obtained, if the tolerance in the previous step was 0, i.e. $A_{min} = A_{max} = \bar{A}$ in

Fig. 5. Territories after 9 iterations (splits resolved).

(4.5). Otherwise the allocation procedure uses up the tolerance for globally compensating customer distribution, resulting in territory sizes which are all at the lower limit in the north and at the upper limit in the south, and leaving no free scope for resolving the splits. Also, the splits are easier to resolve after several location iterations than after the first allocation with the initial center locations, as Table 2 shows. We therefore used the tolerance 0 in the location–allocation stage and then increased the tolerance step by step, even for the local assignment heuristic. The advantage of this procedure is that firstly the splits are resolved as far as possible under tighter restrictions and the larger tolerances are used only in a small number of the cases, and secondly management sees the trade-off between the tolerance and the number of unresolved splits.

In all cases, the local assignment heuristic gave better results than the heuristic based on the transportation problem. We also tried to apply the former *after* the latter. The result was nearly

Table 2. Number of unresolved splits

	Initial allocation		9 location iterations	
	Heuristic of section		Heuristic of section	
Tolerance (%)	5.2	5.3	5.2	5.3
0	161	161	156	156
2	122	96	105	85
4	100	75	88	56
6	86	61	66	33
8	77	49	56	29
10	69	40	48	26

the same as for the local assignment heuristic alone, with the same number of splits left and differences in two territories only.

A refinement of the model, suggested by the analysis of the first results, was the consideration of geographical obstacles, such as the lower course of the rivers Elbe and Weser without bridges. But instead of modifying the calculation of the distances as usually, we rather chose a simpler way for involving obstacles: the user may define a list of forbidden assignments between certain areas and initial territorial centers, very short in our case, which causes the corresponding edges to be suppressed when the network of (4.5) is generated.

The ease of the total procedure and the final result satisfied management perfectly. The remaining manual work is less than expected and extremely light compared with previous experience. The 26 final splits consist of 23 areas with 2 adjacent territories, 1 area with 3 adjacent territories and 1 chain of 2 areas and 3 territories. Hence, the split areas, whose size ranges between 18.5 and 112.8 with a mean of 59, are nearly independent and can be subdivided easily.

The above calculations have served as a successful test of the whole system. However, they also revealed shortcomings in customer data, in particular wrong or missing postal codes. In the meantime, the data have been revised and the system has been used for fixing new territories definitively.

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