Realistic Mathematics Education[1]

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What is Realistic Mathematics Education?

Realistic Mathematics Education – hereafter abbreviated as RME – is a domain-specific instruction theory for mathematics, which has been developed in the Netherlands. Characteristic of RME is that rich, 'realistic' situations are given a prominent position in the learning process. These situations serve as a source for initiating the development of mathematical concepts, tools and procedures and as a context in which students can in a later stage apply their mathematical knowledge, which then gradually has become more formal and general, and less context-specific.

Although 'realistic' situations in the meaning of 'real-world' situations are important in RME, 'realistic' has a broader connotation here. It means students are offered problem situations which they can imagine. This interpretation of 'realistic' traces back to the Dutch expression 'zich REALISEren', meaning 'to imagine'. It is this emphasis on making something real in your mind that gave RME its name. Therefore, in RME, problems presented to students can come from the real world, but also from the fantasy world of fairy tales, or the formal world of mathematics, as long as the problems are experientially real in the student's mind.

The onset of RME

The initial start of RME was the founding in 1968 of the Wiskobas ('mathematics in primary school') project initiated by Edu Wijdeveld and Fred Goffree, and joined not long after by Adri Treffers. In fact, these three mathematics didacticians created the basis for RME. In 1971, when the Wiskobas project became part of the newly-established IOWO Institute, with Hans Freudenthal as its first director, and in 1973 when the IOWO was expanded with the Wiskivon project for secondary mathematics education, this basis received a decisive impulse to reform the prevailing approach to mathematics education.

In the 1960s, mathematics education in the Netherlands was dominated by a mechanistic teaching approach; mathematics was taught directly at a formal level, in an atomized manner, and the mathematical content was derived from the structure of mathematics as a scientific discipline. Students learned procedures step-by-step with the teacher demonstrating how to solve problems. This led to inflexible and reproduction-based knowledge. As an alternative for this mechanistic approach, the 'New Math' movement deemed to flood the Netherlands. Although Freudenthal was a strong proponent of the modernization of mathematics education, it was his merit that Dutch mathematics education was not affected by the formal approach of the New Math movement and that RME could be developed.

[1] Van den Heuvel-Panhuizen, M., & Drijvers, P. (in press). Realistic Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. xxx-xxx). Dordrecht, Heidelberg, New York, London: Springer.

Freudenthal's guiding ideas about mathematics and mathematics education

Hans Freudenthal (1905-1990) was a mathematician born in Germany who in 1946 became a professor of pure and applied mathematics and the foundations of mathematics at Utrecht University in the Netherlands. As a mathematician he made substantial contributions to the domains of geometry and topology.

Later in his career, Freudenthal (1973, 1991) became interested in mathematics education and argued for teaching mathematics that is relevant for students and carrying out thought experiments to investigate how students can be offered opportunities for guided re-invention of mathematics.

In addition to empirical sources such as textbooks, discussions with teachers and observations of children, Freudenthal (1983) introduced the method of the didactical phenomenology. By describing mathematical concepts, structures, and ideas in their relation to the phenomena for which they were created, while taking into account students' learning process, he came to theoretical reflections on the constitution of mental mathematical objects, and contributed in this way to the development of the RME theory.

Freudenthal (1973) characterized the then dominant approach to mathematics education in which scientifically structured curricula were used and students were confronted with ready-made mathematics as an 'anti-didactic inversion'. Instead, rather than being receivers of ready-made mathematics, students should be active participants in the educational process, developing mathematical tools and insights by themselves. Freudenthal considered mathematics as a human activity. Therefore, according to him, mathematics should not be learned as a closed system, but rather as an activity of mathematizing reality and if possible even that of mathematizing mathematics.

Later, Freudenthal (1991) took over Treffers' (1987) distinction of horizontal and vertical mathematization. In horizontal mathematization, the students use mathematical tools to organize and solve problems situated in real-life situations. It involves going from the world of life into that of symbols. Vertical mathematization refers to the process of reorganization within the mathematical system resulting in shortcuts by using connections between concepts and strategies. It concerns moving within the abstract world of symbols. The two forms of mathematization are closely related and are considered of equal value. Just stressing RME's 'real-world' perspective too much may lead to neglecting vertical mathematization.

The core teaching principles of RME

RME is undeniable a product of its time and cannot be isolated from the worldwide reform movement in mathematics education that occurred in the last decades. Therefore, RME has much in common with current approaches to mathematics education in other countries. Nevertheless, RME involves a number of core principles for teaching mathematics which are inalienable connected to RME. Most of these core teaching principles were articulated originally by Treffers (1978), but were reformulated over the years, including by Treffers himself.

In total six principles can be distinguished.

- The *activity principle* means that in RME students are treated as active participants in the learning process. It also emphasizes that mathematics is best learned by doing mathematics, which is strongly reflected in Freudenthal's interpretation of mathematics as a human activity, as well as in Freudenthal's and Treffers' idea of mathematization.

- The *reality principle* can be recognized in RME in two ways. First, it expresses the importance that is attached to the goal of mathematics education including students' ability to apply mathematics in solving 'real-life' problems. Second, it means that mathematics education should start from problem situations that are meaningful to students, which offers them opportunities to attach meaning to the mathematical constructs they develop while solving problems. Rather than beginning with teaching abstractions or definitions to be applied later, in RME, teaching starts with problems in rich contexts that require mathematical organization or, in other words, can be mathematized and put students on the track of informal context-related solution strategies as a first step in the learning process.

- The *level principle* underlines that learning mathematics means students pass various levels of understanding: from informal context-related solutions, through creating various levels of shortcuts and schematizations, to acquiring insight into how concepts and strategies are related. Models are important for bridging the gap between the informal, context-related mathematics and the more formal mathematics. To fulfill this bridging function, models have to shift – what Streefland (1993) called – from a 'model of' a particular situation to a 'model for' all kinds of other, but equivalent, situations.

  Particularly for teaching operating with numbers, this level principle is reflected in the didactical method of 'progressive schematization' as it was suggested by Treffers and in which transparent whole-number methods of calculation gradually evolve into digit-based algorithms.

- The *intertwinement principle* means mathematical content domains such as number, geometry, measurement, and data handling are not considered as isolated curriculum chapters, but as heavily integrated. Students are offered rich problems in which they can use various mathematical tools and knowledge. This principle also applies within domains. For example, within the domain of number sense, mental arithmetic, estimation and algorithms are taught in close connection to each other.

- The interactivity principle of RME signifies that learning mathematics is not only an individual activity but also a social activity. Therefore, RME favors whole-class discussions and group work which offer students opportunities to share their strategies and inventions with others. In this way students can get ideas for improving their strategies. Moreover, interaction evokes reflection, which enables students to reach a higher level of understanding.

- The *guidance principle* refers to Freudenthal's idea of 'guided re-invention' of mathematics. It implies that in RME teachers should have a pro-active role in students' learning and that educational programs should contain scenarios which have the potential to work as a lever to reach shifts in students' understanding. To realize this, the teaching and the programs should be based on coherent long-term teaching-learning trajectories.

Various local instruction theories

Based on these general core teaching principles a number of local instruction theories and paradigmatic teaching sequences focusing on specific mathematical topics have been developed over time. Without being exhaustive some of these local theories are mentioned here. For example, Van den Brink (1989) worked out new approaches to addition and subtraction up to twenty. Streefland (1991) developed a prototype for teaching fractions intertwined with ratios and proportions. De Lange (1987) designed a new approach to teaching matrices and discrete calculus. In the last decade, the development of local instruction theories was mostly integrated with the use of digital technology as investigated by

Drijvers (2003) with respect to promoting students' understanding of algebraic concepts and operations. Similarly, Bakker (2004) and Doorman (2005) used dynamic computer software to contribute to an empirically grounded instruction theory for early statistics education and for differential calculus in connection with kinematics respectively.

The basis for arriving at these local instruction theories was formed by design research, as elaborated by Gravemeijer (1994), involving a theory-guided cyclic process of thought experiments, designing a teaching sequence and testing it in a teaching experiment, followed by a retrospective analysis which can lead to necessary adjustments of the design.
Last but not least, RME also led to new approaches to assessment in mathematics education (De Lange 1987; Van den Heuvel-Panhuizen 1996).

Implementation and impact

In the Netherlands, RME had and still has a considerable impact on mathematics education. In the 1980s, the market share of primary education textbooks with a traditional, mechanistic approach was 95% and the textbooks with a reform-oriented approach – based on the idea of learning mathematics in context to encourage insight and understanding – had a market share of only 5%. In 2004, reform-oriented textbooks reached a 100% market share and mechanistic ones disappeared. The implementation of RME was guided by the RME-based curriculum documents including the so-called 'Proeve' publications by Treffers and his colleagues, which were published from the late 1980s, and the TAL teaching-learning trajectories for primary school mathematics, which have been developed from the late 1990s.

A similar development can be seen in secondary education, where the RME approach also influenced textbook series to a large extent. For example, Kindt (2010) showed how practicing algebraic skills can go beyond repetition and be thought-provoking. Goddijn et al. (2004) provided rich resources for realistic geometry education, in which application and proof go hand in hand.

Worldwide, RME is also influential. For example, the RME-based textbook series 'Mathematics in Context' has a considerable market share in the USA. A second example is the RME-based 'Pendidikan Matematika Realistik Indonesia' in Indonesia.

A long-term and ongoing process of development

Although it is now some forty years from the inception of the development of RME as a domain-specific instruction theory, RME can still be seen as work in progress. It is never considered a fixed and finished theory of mathematics education. Moreover, it is also not a unified approach to mathematics education. That means that through the years different emphasis was put on different aspects of this approach and that people who were involved in the development of RME –mostly researchers and developers of mathematics education, and mathematics educators from within or outside the Freudenthal Institute – put various accents in RME. This diversity, however, was never seen as a barrier for the development of RME, but rather as stimulating reflection and revision, and so supporting the maturation of the RME theory. This also applies to the current debate in the Netherlands (see Van den Heuvel-Panhuizen 2010) which voices the return to the mechanistic approach of four decades back. Of course, going back in time is not a 'realistic' option, but this debate has made the proponents of RME more alert to keep deep understanding and basic skills more in balance in future developments of RME and to enhance the methodological robustness of the research that accompanies the development of RME.

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