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On students’ conceptions of arithmetic average:  
the case of inference from a fixed total

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There is more to understanding the concept of mean than simply knowing and applying the add-them-up and divide algorithm. In the following, we discuss a component of understanding the mean – inference from a fixed total – that has been largely ignored by researchers studying students understanding of mean. We add this component to the list of types of reasoning needed to understand mean and discuss student responses to tasks designed to elicit this component of reasoning. These responses reveal that inference from a fixed total reasoning is rare even in advance high school students.

Keywords: average; inference; mean; student thinking

1. Introduction

The notion of mean – arithmetic average – is central to the study of statistics. It has important implications in a variety of areas that surround our daily lives, such as meteorology, medicine or agronomics, to mention just a few. It is also an important concept for informed citizens [1].

However, the meaning of mean is not sufficiently understood by students of different ages and of different mathematical sophistication. Unfortunately, many students struggle with reasoning with and about the mean in contexts that differ from a simple implementation of the algorithm. Different ways of understanding the idea of mean and of utilizing it in problem solving situations were noted by researchers [2–4]. Our study builds on these works and extends them by highlighting an additional component – inference from a fixed total – that we view as essential to the conceptual understanding of mean.

1.1. On difficulties and various interpretations

Researchers noted specific difficulties associated with the concept of mean. For example, Strauss and Bichler [5] concluded that students in Grades 4 through 8 experienced difficulties with understanding the properties of mean, such as the sum of deviations from the mean is zero, or that the number representing the mean does not have to correspond to physical reality. Gal et al. [6,7] pointed out the difficulties...
Grade 6 students have in using means to compare two different-sized sets of data. Their participants often compared sets using visual comparison or by finding the total of the data sets, rather than by using some measure of central tendency, such as arithmetic average. Moreover, Pollatsek et al. [8] demonstrated college students’ inability to handle averaging problems that involve weighted means. These difficulties indicated that the concept of mean is much more nuanced and complicated than a simple application of the algorithm.

In studying the development of the concept of mean, researchers noted different levels through which students’ understanding evolves. Mokros and Russell [3] theorized that a prerequisite to students’ ability to view the mean as an object is their ability to view a data set that generates it as a single object. They utilized two types of problems in their research: construction problems that provide the student with a mean and ask him/her to construct or partially construct a set which has that mean, and interpretation problems that involve reasoning about a weighted mean of means. In their exploration of students’ responses to these tasks, Mokros and Russell noted five predominant approaches used by students. These approaches are listed in increasing order of sophistication: average as mode, average as algorithm, average as reasonable, average as midpoint and average as mathematical point of balance. The first two approaches were considered nonrepresentative, while the last three were considered representative, in the sense that they imply the concept of average as representative of a set of values.

Watson and Moritz [4], in their longitudinal investigation of ideas related to average, distinguished the following levels of responses of increasing complexity: preaverage, single colloquial usage, multiple structures, representation and application in complex tasks. The complex tasks used in this study involved the consideration of weighted averages and of open-ended construction tasks in which the solution needed to consider average as a point of balance. Note that the latter corresponds to the upper level in the categorization provided by Mokros and Russell [3].

Cai [1] focused on conceptual understanding of arithmetic average. He argued ‘conceptual understanding of average involves understanding it, both as a statistical idea for describing and making sense of data sets and as a computational algorithm for solving problems’ (p. 96). He demonstrated that while most of the Grade 6 students in his study could successfully apply the algorithm for calculating the average, only about one-half of the participants could solve problems that required finding missing data when the average was known. Cai considered the ability to ‘work backwards’ on the missing value problems as an indicator of conceptual understanding of the averaging algorithm; however, he did not detail what could be involved in the conceptual understanding of ‘statistical idea’.

From the studies of instructional interventions, we derive what such conceptual understanding may entail. In an attempt to foster students’ conceptualization of mean researchers emphasized the ‘balance model’ [9], which envisions mean as similar to a balance scale with a fulcrum located at the mean and the ‘fair share model’ [5]. Furthermore, instructional recommendations focused on fostering students’ ability to use average in problem-solving situations in a variety of contexts [4,10]. In particular, Gal [10] highlighted ‘the distinction between knowing a concept and understanding how to use it’ (p. 97), whereas among the learning experiences advocated for students Watson and Moritz [4] noted ‘to explore open-ended problems requiring the mean algorithm to be reversed and the total seen as an
important link in the process’ (p. 47). These recommendations are consistent with the categorization of levels of understanding, where ‘Application in Complex Tasks’ is on the top of the hierarchy.

Konold and Pollatsek [2] itemized four different contexts for interpretation of average: data reduction, fair share, typical value and signal and noise. They argued that the last interpretation, signal and noise, was ‘the most fundamental conceptual model for reasoning statistically’ (p. 286).

While we agree with the importance of the distinctions and recommendations mentioned above, we add a component of conceptual understanding that was not specifically attended to in prior research – ability to infer the mean from a fixed total. We explain what such inference involves in the following section.

1.2. Focus on inference from a fixed total

We illustrate the idea of ‘inference from a fixed total’ with the following two tasks:

(A) The prize of $1000 was split among five people, so that each person received a different amount. What was the value of the average prize?

(B) The prize of $1000 was split among five people, so that the amount received were 120, 190, 250, 340 and 410. What was the value of the average prize?

Since the total is given, and the number of data points that contribute to this total is known, the specific values in the set of numbers that contribute to this fixed total have no relevance to the solution. When students are able to solve Task A without any additional prompting and to ignore the irrelevant numbers in Task B we say that they have the ability to infer from a fixed total.

Task A may appear as trivial for an expert, which may explain why tasks of this kind are not usually included neither in textbooks nor in the question sets used in research. However, it may present a challenge for a student who has not yet acquired a concept understanding and serve as an indicator of understanding of a conceptual component. Studies of Mokros and Russell [3,11] demonstrated that students classified in the ‘Average as Reasonable’ group believed it was not possible to know the exact average of a set without knowing the value of each quantity in the set. As such, these students would most likely ignore the total value in Task B and apply a complete algorithm starting with summation of particular numbers. In fact, inference from a fixed total requires only a partial application of the algorithm for calculating the mean: skipping the addition step and performing only the division step. However, we suggest, that this ability is an essential component in understanding the statistical idea of mean and we explore it in this study.

2. The study

2.1. Participants

Two groups of high school students participated in the study. The students in the first group (Group A, n = 56) were enrolled in a Grade 12 mathematics course. The students in the second group (Group B, n = 23) were enrolled in a high school calculus course. Neither course was required for graduation in British Columbia (Canada), where the data were collected, at the time of data collection. However, completion of a Grade 12 mathematics course was required for entrance for several
tertiary programmes in Sciences and Engineering. The calculus course, that follows
the Grade 12 mathematics course, is intended to ease the transition to university
calculus by covering much of the material which university calculus entails. As such,
the participants were drawn from the population of students that chose to pursue
their study of mathematics beyond the minimal requirement for high school
graduation, which is a Grade 11 mathematics course. These two groups were not
chosen for direct comparison; they are at different points in their mathematical
careers so such a comparison would be inappropriate. Instead they were chosen to
give a sense of how students at different points in their mathematics careers respond
to inference from a fixed total questions.

2.2. Research questions
How do students describe and exemplify the idea of mean?
How do students approach tasks that could be solved using ‘inference from
a fixed total?’

2.3. The tasks
Figure 1 shows the tasks that were developed to address our research questions.
Task 1 does not simply ask what arithmetic average is, but asks to explain it to a
younger student and to exemplify. We hoped that the mention of a younger student
would elicit personal images of the concept.

Tasks 2 and 3 can be solved immediately without any computation by utilizing
inference from a fixed total. In Task 2, the sum of the angles in a triangle is a
constant, as such the mean value of an angle is $60^\circ$, regardless of the particular angle
values provided in the task. In Task 3, the total area is constant, regardless of the
value of particular sectors, as such the mean value of one of the four sectors is $\pi/4$.
Given that participants in our study were high school students with relatively strong
mathematical backgrounds, we assumed that the formulae for the area of a circle
$(\pi r^2)$ and the sum of the angles in a triangle $(180^\circ)$ were part of their common and
readily accessible knowledge.

3. Results and analysis
3.1. Task 1
Responses to Task 1, which asked students how they would explain mean to a
younger student, were partitioned based on two characteristics. The first was
whether a student’s response explained what a mean is (what-type response) or
just how to find one (how-type response). This coding is documented in Table 1.
The second characteristic was the mode students used to describe the mean algorithm
(verbal, numerical example, abstract symbols, etc.).

With respect to the first characteristic, how-type responses described the
algorithm for finding a mean in some way. In contrast, what-type responses
included a description of what the end result of the algorithm represented.
These involved either some kind of description of central tendency, such as ‘closest
to the middle of a group of numbers’, ‘it’s just the middle’, or ‘approximate middle’,
or a description of typicality like ‘normal value’ or ‘typical number’. These responses
Task 1
How would you explain the idea of mean (arithmetic average) to a younger student?
Please write down your explanation and provide an example.

Task 2
The measures of angles of the two triangles are shown below.
What is the mean (arithmetic average) measure of these six angles? __________
Explain your conclusion.

Task 3
The circle is divided into 4 sectors, as shown in the picture. The numbers indicate the measures of the angles. The area of each sector is proportional to the area of the circle and can be calculated using the formula:
Sector area = \( \frac{\alpha}{360} \pi R^2 \)
where \( \alpha \) is the angle of the sector and \( R \) is the radius.

What is the mean (arithmetic average) area of the 4 sectors, if \( R=1 \). Explain your solution

Figure 1. The tasks.

Table 1. Frequency of occurrence of different types of responses to Task 1.

<table>
<thead>
<tr>
<th></th>
<th>Group A (n = 56)</th>
<th>Group B (n = 23)</th>
</tr>
</thead>
<tbody>
<tr>
<td>How-type answer</td>
<td>35 (63%)</td>
<td>11 (48%)</td>
</tr>
<tr>
<td>What-type answer</td>
<td>6 (11%)</td>
<td>8 (34%)</td>
</tr>
<tr>
<td>Nonsensical answer</td>
<td>13 (23%)</td>
<td>2 (9%)</td>
</tr>
<tr>
<td>Blank</td>
<td>2 (3%)</td>
<td>2 (9%)</td>
</tr>
</tbody>
</table>

indicate a meaning for the concept of mean, rather than just a description for how to calculate it. What-type responses were generally also accompanied by descriptions of the algorithm for calculating mean. However, the presence/absence of these did not influence whether a response was categorized as what-type. All but two of the responses that included a cogent description of mean also contained a description of the algorithm.

Figure 2 illustrates a response that was coded as what-type. It describes mean in terms of central tendency – ‘represents the middle of something’. It also
supplements this by describing verbally how one would find this middle. Of note is
that this student chooses to use a context to describe the mean algorithm; it is a mean
of something rather than just an abstract meaningless calculation. Contextualization
of explanations appeared in six what-type responses, but appeared only once in
how-type responses.

As shown in Table 1, it was much more common for Group B students to provide
what-type solutions (8 out of 23, 34% of responses) compared to Group A students
(6 out of 56, 11% of responses), although in both groups the majority of students
simply gave a description of the mean algorithm. It is, however, promising that not
all students described mean as simply an algorithm, this points to students having
at least some exposure to mean as more than an algorithmic procedure.

It should be noted that students who only wrote synonyms for mean, such as ‘it’s
just the average’ were not lumped into either the ‘what’ or the ‘how’ category since
their descriptions did not indicate any knowledge of mean apart from knowing the
word had synonyms. These responses were instead lumped into a nonsense category
along with responses such as ‘I don’t know’.

Also coded was how the algorithm was described. Each response was classified as
some combination of verbal, numerical or abstract. If an example was used, it was
noted whether the example was contextualized within a specific situation or consisted
of arbitrary numbers not tied to a context. Verbal explanations, which were found
in the majority of responses (80%, 64 out of 79 total responses), describe the
algorithm in words. For example, ‘add up all the numbers and divide by the number
of numbers’. Numerical examples work through the algorithm with some specific
set of numbers. This was either associated with a context such as calculating a class’s
mean exam grade or simply performed on an arbitrary set of numbers. Abstract
explanations were rare. These types of examples consist of some generalized
notation, which describes the algorithm. For example, \[ \frac{x_1 + x_2 + \ldots + x_n}{n} \]. There did not
appear to be any relationship between how the algorithm was described and whether
or not the student provided a why or a how-type explanation.

3.2. Tasks 2 and 3

Tasks 2 and 3 were coded for their correctness. The correct solutions were then
subcategorized according to whether students used inference from a fixed total or
strictly applied the algorithm (Table 2). No students who had incorrect solutions
showed evidence of inference from a fixed total type thinking. Several students
showed evidence of inference from a fixed total thinking in only one of the
tasks. However, these students always showed evidence of inference from a fixed
total on the triangle problem (Task 2) and did not show evidence of such thinking.
on the circle problem (Task 3). No students showed evidence of inference from a fixed total on only Task 3. As shown in Table 2, in both Groups A and B, the majority of participants did not demonstrate evidence of inference from a fixed total thinking. Only 25% (14 out of 56) of Group A and 43% (10 out of 23) of Group B students demonstrated inference from a fixed total in some form. This points to a concerning lack of such thinking in advanced high school mathematics students.

Figure 3 exemplifies a student response in which inference from a fixed total thinking is used on both Tasks 2 (Figure 3a) and 3 (Figure 3b). Note that the student started Task 3 by attempting to use standard computational methods and changed strategies halfway through the problem solving process, after she made the observation that the total would be the same regardless of how the circle is split. Once this observation is made, the solution requires very little calculation. In her response to Task 2 she also notes that the result would hold for any triangle, an observation that requires inference from a fixed total thinking.

Table 2. Categorization of responses on Tasks 2 and 3.

<table>
<thead>
<tr>
<th></th>
<th>Group A ( (n = 56) )</th>
<th>Group B ( (n = 23) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blank</td>
<td>3 (5%)</td>
<td>1 (4%)</td>
</tr>
<tr>
<td>Incorrect</td>
<td>15 (27%)</td>
<td>6 (26%)</td>
</tr>
<tr>
<td>Used the standard algorithm</td>
<td>24 (43%)</td>
<td>6 (26%)</td>
</tr>
<tr>
<td>Inference from a fixed total only on Task 2</td>
<td>8 (14%)</td>
<td>14 (25%)</td>
</tr>
<tr>
<td>Inference from a fixed total on both Tasks 2 and 3</td>
<td>6 (11%)</td>
<td>6 (26%)</td>
</tr>
<tr>
<td></td>
<td>6 (11%)</td>
<td>10 (43%)</td>
</tr>
</tbody>
</table>
Figure 4 exemplifies a student response in which the standard algorithm is used. In this solution, the student finds a decimal approximation of the area of each section. Each of these is rounded for ease of use and then the set of approximations is run through the mean algorithm. Due to rounding errors the student gets a total of 3.15, which he does not recognize as $\pi/4$, the area of the circle. The final solution is presented in decimal form, which gives the student no indication of how his answer relates to the actual area of the circle. By implementing the standard algorithm on decimal approximations he has stripped the final answer of its meaning by camouflaging its relation to the actual area.

It is particularly telling to compare the standard algorithm solution shown in Figure 4 to the solution that demonstrated inference from a fixed total shown in Figure 3. The response in Figure 3 demonstrates a deep understanding of the components involved in Task 3 and how they interrelate. In contrast, the response in Figure 4 is no more than a routine calculation. In it, the solution, although correct, has been stripped of its meaning.

We are rather surprised by the number of students (12) who exemplified inference from a fixed total on Task 2 but not on Task 3. It could be the case that the total sum of the angles of a triangle is more evident for students as a numerical constant than the sum of the areas of sectors that compose a circle. However, further research is needed to investigate how inference from a fixed total thinking is influenced by context of a particular task.

4. Discussion

Shaughnessy [12] argued that students’ understanding of the average concept was the add-them-all up and divide algorithm, because the computational algorithm was all they were ever taught. However, most research confirming this finding was carried out with upper elementary or middle school students. Our study was carried out with Grade 12 students who were enrolled in mathematics courses beyond the minimal requirements for graduation. As such, the participants had relatively strong mathematics backgrounds. Despite this relative strength, the majority of participants focused on calculating mean when asked to explain the idea (Task 1) and carried out the algorithm when an immediate solution was attainable via inferring from a fixed total.
However, taking an optimistic look at our findings, we note that 14 out of 79 participants (18%) in our study (six from Group A and eight from Group B) indicated some sort of meaning behind the algorithmic calculation of mean. This is evident in their description of what mean represents in the responses to Task 1. Furthermore, 24 participants (14 (25%) from Group A and 10 (43%) from Group B) used inference from a fixed total in at least one of the Tasks 2 and 3, and 12 participants used inference from a fixed total on both tasks. It is interesting to note that the 14 students who described mean without relying solely on the algorithm were not a subset of the 24 participants who implemented inference from a fixed total. In fact, only three students were in the intersection of these two sets. To reiterate, the students who were able to see the underlying structure of the problems they were given and use this to find a solution were not necessarily the students who had what-type explanations of what a mean is. This leads us to the conclusion that inference from a fixed total may be connected to a conceptual understanding of the algorithm rather than on a conceptual understanding of the idea of mean itself. Or, more accurately, ability to infer mean from a fixed total does not depend on the ability to describe this notion to others as more than an algorithm.

Konold and Pollatsek [2] summarized the types of school tasks that involve averages and averaging. In addition to popular tasks of finding the average of a group of numbers, they mention tasks that assess the understanding of mean as a balance point, and of the properties that mean lies somewhere within the range of scores and that it does not need to correspond to a particular value in the set. These do not include tasks that can be approached by inference from a fixed total. Further, we also did not find tasks that assessed inference from a fixed total in prior research. We argue that such tasks may serve to reinforce and enrich students’ understanding of mean as well as to examine a particular component of their understanding.

Our contribution is in adding a component – inference from a fixed total – to the list of properties connected to the understanding of mean provided by Strauss and Bichler (1988), and in designing tasks that could elicit this component in students’ solution strategies. Furthermore, utilizing these tasks in our research, our findings demonstrate that inference from a fixed total reasoning is rare even in fairly advanced students.

We acknowledge that the limitation of our study is in relying on written data only, and that some of the results may be interpreted as students’ lack of cooperation (they were assured that the questionnaire had no influence on their course mark) or as their desire to fulfil perceived expectations (e.g. by carrying out the algorithm explicitly). With respect to the latter, we wondered whether the availability of numbers guided students’ solutions in a computational direction. As such, in our informal follow-up study we used a task similar to Task 3: the same diagram and prompt to find the average area of the sectors accompanied the task, however the particular measures of angles were deleted.

This modified version of Task 3 was presented in a clinical interview setting to eight Grade 12 students of average ability. Seven out of eight stated that it was impossible to solve the problem without knowing the specific areas of the sectors. These students stated it was a trick question and that ‘You can’t solve an average problem without stuff to average’. The eighth student, who solved the task correctly, struggled with the task for several minutes. In the process of working on it, he tried several different possible ways of splitting up the circle before realizing why the
solution was invariant to such changes. His inference from the fixed total was far from immediate. Further research needs to investigate the idea of inference from a fixed total in different settings, such as realistic situations and those free of the geometry context. Moreover, the development of learning trajectories that encourage the formation of such reasoning should be an additional focus of follow-up research and instructional development.

References