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Construction of the vector space concept from the viewpoint of APOS theory☆

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ABSTRACT

We apply APOS theory to propose a possible way that students might follow in order to construct the vector space concept. We describe the mental mechanisms and constructions that might take place when students are learning this concept. We then report on a study that we performed with 10 undergraduate mathematics students through the application of a questionnaire and an interview. In this paper we focus on the coordination between the two operations that form part of the vector space structure and the relation of the vector space schema to other concepts such as linear independence and binary operations.

1. Introduction

The vector space concept which is of great importance in linear algebra has received attention from researchers in different countries. French researchers (Dorier, Robert, Robinet, Rogalski) talk about the obstacle of formalism. This obstacle occurs when students try to manipulate numbers, vectors, equations, coordinates, etc. when they submerge "under an avalanche of new words, symbols, definitions and theorems". These authors conclude that "for the majority of students, Linear Algebra

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is nothing more than a catalogue of very abstract notions that they are never able to imagine". On the other hand, they warn that it is extremely difficult to find situations at their level where the concepts of linear algebra would play the role of tools to be used in solving problems. This fact is related to the unifying and generalizing nature of this subject, and these authors suggest alternative approaches such as "meta-lever" to introduce and develop more abstract concepts of Linear Algebra [1,2]. Other research points out to difficulties that students have when learning this concept [3].

2. Theoretical framework: APOS theory

The use of APOS theory to explain the construction of linear algebra concepts is recent [4–7], although this theoretical approach has been used successfully in research concerning the learning of mathematical concepts in Calculus, Analysis, Abstract Algebra, Discrete Mathematics and Logic. APOS theory is interested in mental constructions that students make when they are learning a mathematical concept. When using this theory, the researchers first make a description of a model that might explain the way that students would follow in order to make the proposed constructions.

This model is known as the *genetic decomposition* [8] and consists of mental constructions (Actions, Processes, Objects, Schemas) and mental mechanisms (such as assimilation, interiorization, encapsulation, de-encapsulation, coordination) put together in a way to explain the learning of the concept in question. It should be emphasized that the genetic decomposition is given in terms of cognitive constructions, and not in terms of mathematical results. We now briefly explain some of these notions that we have used in our research.

We talk about an *action conception* of a concept when the individual can perform calculations and transformations of mathematical objects as a result of external stimuli, such as plugging in numbers for variables in a formula. (S)he can also perform multiple step algorithms, where each step is triggered by the previous one.

When the individual reflects about these actions, (s)he can run through the steps in her/his mind without having to perform them explicitly. In this case we say that the actions have been *interiorized* and the individual possesses a *process conception*. Two or more processes can be *coordinated* to form a new process.

When the need arises to perform transformations on these processes, the individual *encapsulates* them into objects and now can apply actions on these newly constructed entities. In this case (s)he shows an *object conception* of the concept in question. When necessary, the object can be *de-encapsulated* for the individual to have access to the underlying processes.

Finally objects, processes and actions related to the concept in question form a coherent structure called a *schema* which can be invoked in order to resolve problem situations. A new object can be *assimilated* by an existing schema; this way the reach of the schema is expanded in order to include new objects. According to Piaget schema development passes through three levels: Intra, Inter and Trans. At the *Intra* level the newly constructed object is present, together with other objects and processes, but at this stage the individual is not aware of the relationships that might exist between them. At the *Inter* level these relationships start to be present and *Trans* level is characterized by being aware of the complete structure and being able to decide whether a given situation can be resolved by that particular schema.

3. Previous work concerning possible genetic decompositions of the vector space concept

In 2002, some members of RUMEC (Research in Undergraduate Mathematics Education Society) published through Internet teaching materials in Linear Algebra that were based on APOS theory [9]. These materials included activities and exercises using the computer programming language ISETL, as well as traditional problems and discussion about each one of the introductory topics in Linear Algebra. Although the materials did not include them explicitly, each topic had an accompanying genetic decomposition, including the concept of vector space [10].

The 3-page report on vector space and subspace concepts described the activities that would be included in the teaching materials, and provided a genetic decomposition for the same concepts, referring to these activities. The document starts by saying that "The concept of vector space is a schema that is constructed by coordinating the three schemas of set, binary operation, and axiom". After describing each of these schemas and providing the activities to be given to the students, the following genetic decomposition is presented (we leave out the part concerning subspaces, as it is beyond the scope of this paper).

> RELATIONSHIP BETWEEN THE THREE SCHEMAS IN DEVELOPMENT OF THE VECTOR SPACE SCHEMA
> The vector space concept involves three schemas, set, binary operation, and axiom. In constructing the **func is_vector_space**, students are led to make the necessary de-encapsulations and coordinations. In particular, the **func** is a boolean-valued function that accepts a set V, a field of scalars $\mathbf{K}$, an operation defined on V, va, and an operation defined on the pair $(\mathbf{K}, V)$, tests whether the system $(\mathbf{K}, V, \text{va}, \text{sm})$ satisfies all ten axioms, and returns true, if all ten axioms are satisfied, and false, otherwise. In order to apply the axiom schema, the set and binary operation schemas need to be thematized into objects. When the axiom schema is applied, the set and binary operation schemas are de-encapsulated and coordinated with each axiom. The ten instances of this coordination are coordinated into a single process that establishes whether the system constitutes a vector space.
> The vector space schema is thematized to form an object to which actions can be applied. Examples of such actions may include determining whether a given system $(\mathbf{K}, V, \text{va}, \text{sm})$ defines a vector space. Students apply **is_vector_space** to the 12 systems to which they applied the axioms individually [10].

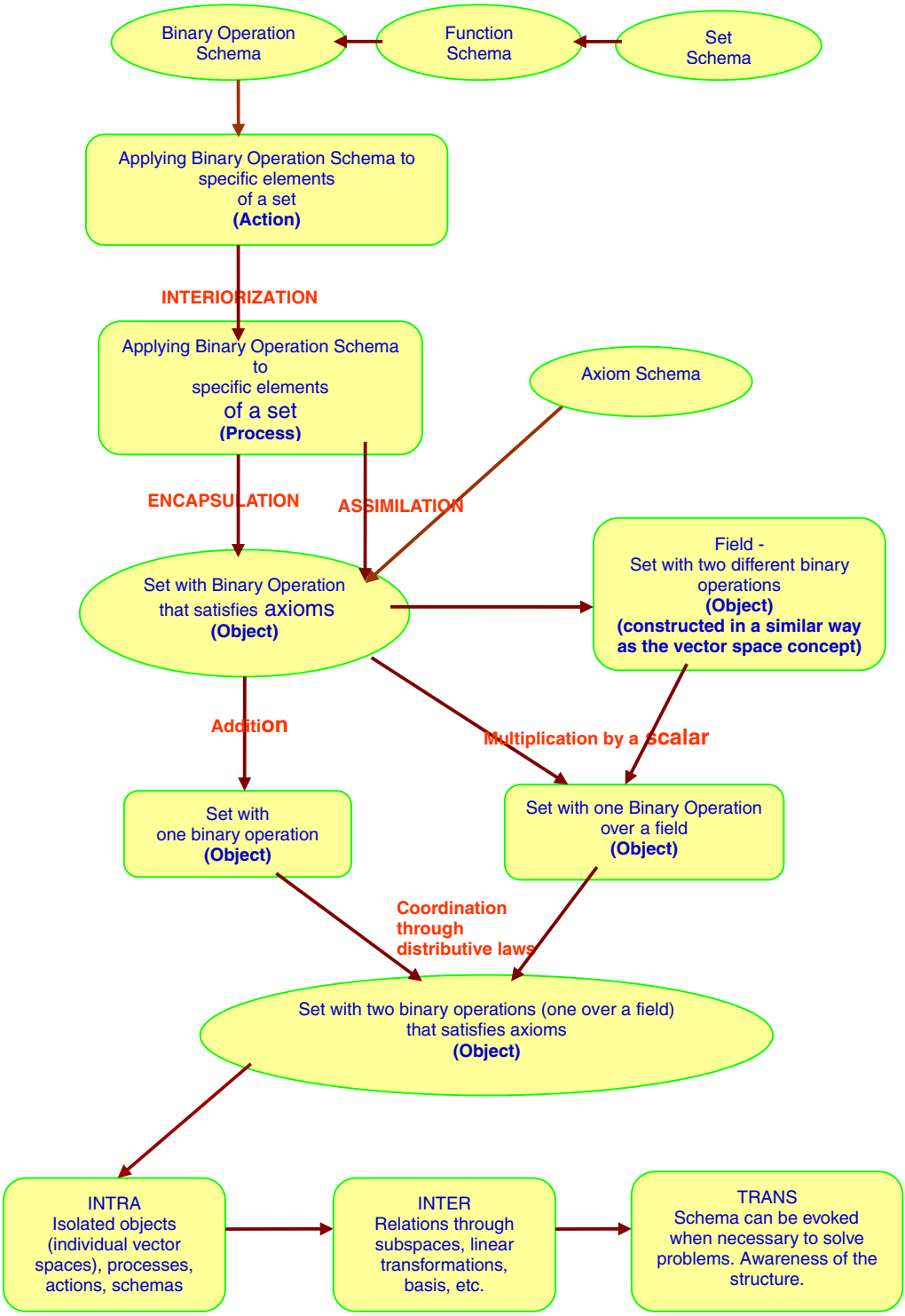
This genetic decomposition provides a good example of how APOS theory cannot be separated from teaching and pedagogical approaches. However the provided genetic decomposition (which was offered as an aide in the preparation of the teaching materials) is somewhat compact and relies heavily on the proposed computer activities. Later, Trigueros and Oktaç [4] presented a generalized version of this genetic decomposition solely in terms of the mental constructions and mechanisms that are implied, keeping the focus on the coordination of the three schemas. Afterwards, based on this genetic decomposition with small modifications, empirical investigation was performed using interviews [6]. Although the results indicated that the constructions predicted by the genetic decomposition seemed to be present in student learning, there were some aspects of vector space theory that were not taken into account in earlier genetic decompositions, such as the coordination between the two binary operations that conformed the vector space structure (which is the focus of this paper). Another point is that the schema development in terms of the triad (Intra–Inter–Trans) was not a part of these previous research projects. Furthermore we put special emphasis on the field concept as part of our genetic decomposition, as empirical results [6] suggest that there are a lot of difficulties related to its connection to the vector space structure.

4. A proposed genetic decomposition of the vector space concept

In what follows, we propose a genetic decomposition based on our experience as teachers and learners of this topic, and previous research results available to us as described above. First we present it as a diagram, and then we explain the components involved.

According to this genetic decomposition in order to construct the concept of vector space a student starts by activating the constructions that (s)he had already made about sets and binary operations.[1] This implies that the student applies a specific binary operation to specific pairs of elements of a set, as an action. That is, given two elements of a specific set and a specific binary operation, (s)he can find the resulting element. This action is interiorized into a process which implies thinking about what the binary operation does to all elements of a set, in a general manner. Then this process is encapsulated into an object to which the individual can apply actions. At this point the object "set with binary operation" can be assimilated by the axiom schema (which also contains quantifiers) to give place to

[1] We are using the expression "binary operation" in a general sense. We are aware that in a scalar multiplication the operated elements do not come from the same set.


Binary Operation Schema
Function Schema
Set Schema
Applying Binary Operation Schema to specific elements of a set (Action)
INTERIORIZATION
Applying Binary Operation Schema to specific elements of a set (Process)
Axiom Schema
ENCAPSULATION
ASSIMILATION
Set with Binary Operation that satisfies axioms (Object)
Field - Set with two different binary operations (Object) (constructed in a similar way as the vector space concept)
Addition
Multiplication by a scalar
Set with one binary operation (Object)
Set with one Binary Operation over a field (Object)
Coordination through distributive laws
Set with two binary operations (one over a field) that satisfies axioms (Object)
INTRA Isolated objects (individual vector spaces), processes, actions, schemas
INTER Relations through subspaces, linear transformations, basis, etc.
TRANS Schema can be evoked when necessary to solve problems. Awareness of the structure.

a new object that is a set with a binary operation that satisfies axioms. The student is able to verify if all the given axioms are satisfied or if there are some that fail. Here the field plays an important role and it is possible to define an operation on the set over a field (this is possible since the student has constructed the set as an object so that new operations can be defined on it). The objects that are sets with two kinds of operations (addition and multiplication by a scalar) can be coordinated through the related processes and the vector space axioms that involve both operations, to give rise to a new object that can be called a vector space. This implies de-encapsulation of the two objects in order to have access to the processes from which they resulted. Then the coordination between these two processes takes place through the distributive laws which involve the operation of addition and the operation of multiplication by a scalar.

At the Intra level the object of vector space stays isolated from other actions, processes, objects and schemas. For example the student can verify different sets as being vector spaces or not, but does not see the vector space structure inherent in all of them. At the Inter level the object of vector space starts having relationships with other concepts such as subspace, linear transformations, basis, etc. When the student reflects upon these relations, through synthesis they can be recognized as part of a whole structure that makes up a vector space schema. This implies that the Trans level is reached and the student can recognize and work with non-standard examples of vector spaces and can invoke her/his schema when needed.

5. Methodology

In order to test the viability of our genetic decomposition we prepared research instruments (a questionnaire and a semi-structured interview); each question was designed to test specific mental constructions (some of the questions were used in a previous study [6]. The questionnaire was applied to a group of six mathematics students who were in their fourth semester and who were taking a second course in Linear Algebra. The results of the analysis of the questionnaire helped shaping the interview questions, which was applied to these six students and four others who were more advanced in their studies (they were in their eighth semester).

We now give some examples of the interview questions, state our purpose in asking them and present results from the interviews (in the extracts that we provide students are numbered S1 through S10 and the interviewer is I).

6. Some examples from the interview

We were particularly interested in observing how and to what extent students were able to coordinate the processes related to the two operations of a vector space, through the distributivity axioms. Question 7 of the interview had the purpose of checking what kinds of strategies students used in looking for the second operation on a given set, so that it becomes a vector space:

6.1. Question 7

$\mathbf{R} - \{0\}$ is an abelian group with the ADDITION operation defined as follows:

$$\text{ADDITION: } x + y =: x \cdot y, \text{ where } x, y \in \mathbf{R} - \{0\}.$$

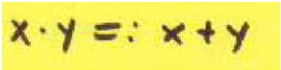
Define the other operation, MULTIPLICATION BY A SCALAR over a field K so that $\mathbf{R} - \{0\}$ becomes a vector space over K, with those two operations.

At the beginning of the interview that corresponds to this question S6 repeats the question a few times to himself and then the interview continues as follows:

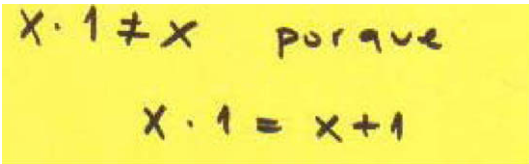
S6: But here I don't understand much, why does it say "define the other operation"?
I: Yes, because it's a vector space. How many operations does it have?
S6: No, no, no, that's fine, but. . . I don't know what it. . ., I mean. . . I mean, let's see.
I: Just tell me.

S6: If there isn't any I have to say it.
I: Exactly, if you think there isn't any.
S6: I mean, basically it is asking if the operation can be defined or not, but here it is asking to define it.
I: To define it, but if you think it cannot be done you can explain why not.
S6: May be. . . may be I can define it.
I: Go ahead.

S6 writes the following:



S6: Perhaps, eh, I would have to see if it satisfies the. . . How are they called? The properties, but with this they are not satisfied.
I: Which ones are not satisfied?, How do you argue to show that they are not satisfied?
S6: x times one is different from x
I: OK, good.
S6: x times one, with this definition would be $x + 1$.



I: So you cannot take that operation, you would have to think of another one.
S6: Can I pass to another question?

We can observe that during the interview S6 does not think about any of the vector space axioms where both operations appear together. In his mind they are two separate processes and there are no signs of coordination between the two.

Another student S5 first tries to use the usual multiplication as the operation that he needs to define. However he realizes that $(x + y)z$ (which is equal to xyz) does not give the same result as $xz + yz(xzyz)$ with the way the two operations are defined. At that point he starts thinking that perhaps it is not possible to define such an operation, although he continues considering other operations. Next he tries to take αx as $\alpha + x$, but he discards it quickly, as $1x$ does not give x. After spending some more time on this question, he has the following reflection about the relationship between addition and multiplication:

S5: When one multiplies it's another thing, it's not adding several times. At some moment I thought multiplying is like adding several times, I mean that's how they teach it to you when you are small, that 2×2 is $2 + 2$, something like that. So I was saying, if it's multiplication here, what is multiplying many times? Elevate to a power, but it cannot be because I cannot take rational number powers of negative numbers. For example I cannot put together -2 with $1/2$, for example.

The interviewer tries to convince S5 that multiplication is not repeated addition, and seemingly S5 accepts, but he does not look too convinced, and gives his opinion about the question:

S5: Yeah, it's just that I had never seen such an exercise, I mean it's not an exercise, it's like how to relate concepts and things like that. Because it's like, I mean it has more to do with the structures of vector spaces, such as talking about the field, the basis of vector spaces.

I: That's how the questions are.
S5: For example, I don't know of any theorem that talks about this. I don't know of any theorem that says, I don't know, a vector space with such and such field, what relationship does the same space have with the same field of scalars, but with another field of scalars, I don't know such a theorem. I don't know if. . . I think there is, but I don't know. . .
I: But you, with what you know, could come up with an argument that would be useful in answering this.
S5: Yes, but as always there are two things: That they don't exist, but there I would have to provide an argument such as a theorem or something like that, that arrives at a contradiction. But I don't have any, I have very few tools. And the other one is to find it but again, finding it seems to me to be a little complicated.

This student ends up claiming that it's not possible to define such an operation. S5 shows some elements of relationships between the two operations, however the whole coordination is not there yet. His efforts stay at the level of trials and errors, without much success.

In the following extract we see another student, S8, reflecting on the relationship that exists between the two operations.

S8: Yes, let's see. . .I take the addition, which is the multiplication, the addition. Yes, I need an operation, multiplication by a scalar that satisfies the following conditions. Let's see. . .I need. . . The minimum that I need to ask this operation, I need an operation of multiplication by a scalar that satisfies the following. I need. . . For it to make sense. . .if I have α and β in K, yes, because the scalars are from K; so it has to satisfy $\alpha(x+y) = \alpha x + \alpha y$ where x, y in R without the zero. Because that's what they are asking from me. And then $(\alpha+\beta)x = \alpha x + \beta x$, yes, and α and β are in K. Let's see, for this condition to be satisfied, the first one, how was this $\alpha(x+y)$ defined? It's α multiplied by, and "$x+y$" is defined as "x times y". And that's what I need – I will write it down as a question here, it should be equal to $\alpha x + \alpha y$. emmm. . . ah and what would it give me? αx . . . Of course I have to define the operation, I have a problem here.
I: What is the problem?
S8: The problem is the meaning that I am giving to that operation, I still haven't found it. . .
I: That is the problem.
S8: Of course. I have to separate an operation that I don't know how it's defined yet. I mean from the beginning I would need to know. . .Well, that's what they are asking me to do. . . what operation is it so that it makes sense?
I: That's right.
S8: Let's see if I, something occurs to me, let's see. . ...Those are the conditions that it has to satisfy.

At this point the interviewer asks whether there are other conditions that also have to be satisfied, and S8 adds $\alpha(\beta x) = (\alpha\beta)x$ to his list of axioms that have to do with the multiplication by a scalar. He continues thinking about the first condition and talks about separating the sum, and that he has to find a way to accomplish it with the second operation. At some point, thinking about the possibility of defining it as a fixed number, he reflects:

S8: The product would have to be varied, I mean the function would have to be like varied. . . .I mean. . ..αx, it cannot be r where r belongs to $R - \{0\}$, constant. It cannot be that because the sum won't. . .
I: won't work, OK. At least that one has been discarded.
S8: So it cannot be fixed real, so I am thinking of conditioning the. . . And now it is complicated because I don't know the field K, so how to define a. . .

S8 cannot conclude the problem, but his reflections show that the coordination has started to take place (perhaps as a result of this interview). He is able to state the conditions that would have to be satisfied, he is able to discard certain operations realizing what the reasons for it are, and he starts to have an idea of the characteristics of the operation he is looking for.

S1, after reading the question, immediately tries defining the operation as $\alpha rmv = \alpha^v$ where α and v are real numbers. When he realizes that for α^1 to be equal to 1, α has to be 1 and hence one of

the axioms is not satisfied, he continues thinking about other possible operations. He tries $\alpha v = v^{\alpha}$ and checking the properties he shows that what he obtains is a vector space. This student from the beginning shows a good grasp of the kind of operation that he is searching and although he has to try another operation first, the coordination of the processes involved in the two operations lead him to discover a correct operation.

A related question was the following, where the students were asked to think about the possibility of the construction of a vector space which satisfies a certain condition.

6.2. Question 3

Is it possible to have a vector space with only two elements?

S4 writes the following response to this question:

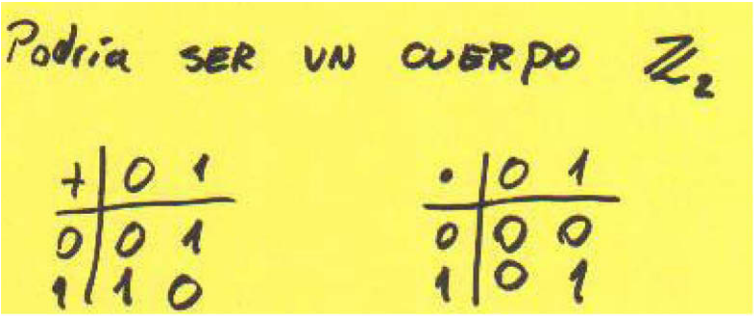
> Yes, since with them I can apply the necessary operations to check that it's a vector space. That is, there exists a vector space with two elements. For example we can say that two lines that generate a plane is a vector space.

In this answer we can see that a weak set schema (which we consider a prerequisite in our genetic decomposition) influences negatively in the construction of the vector space concept.

Another student S6 thinks that it is not possible to have a vector space with two elements, since one would have to be the identity element for addition, and the other element, added to itself, would "come out of the set". This student concludes that it is not possible to have a vector space with a finite number of elements. According to our analysis this student has a weak binary operation schema (which we also consider a prerequisite) and this does not allow him to develop some constructions necessary to have a strong vector space schema.

On the other hand the following interview extract with S3 shows that this student has constructed coherent set and binary operation schemas, as well as field and vector space schemas which are coordinated and used when necessary:

> **S3**: It can be a field with two elements.
> **I**: And what are you thinking about? Please write it down.
> **S3**: A field. . . I think it's called Z sub 2, I'm not sure.
> **S3**: Yeah, where the sum is. . . we take zero and one, there would be these two elements: 0 added to 0 is 0; 0 added to 1 is 1; 1 added to 0 is 1 and 1 + 1 would be 0. . . And the product: 0 times 0 is 0, 0 times 1 is 0, 0 and 1, we need to prove, to see that it satisfies the axioms, the 10 vector space axioms.



S3 goes on to justify that all the 10 axioms are satisfied either because Z_2 is a field or because of the way the operations are defined.

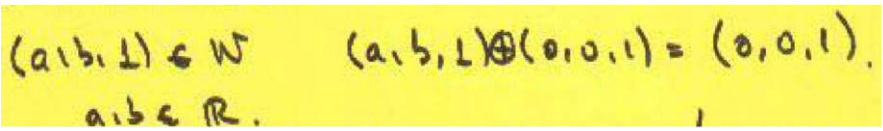
In order to find out how students relate the vector space concept to other concepts, and how the introduction of other concepts (in this case linear independence and basis) might affect the vector space schema, we asked the following question, where the student has to realize that the null vector is not necessarily composed of "zeros" in a vector space.

6.3. Question 8

Let $V = \{(x, y, z) \in R3?x, y, z > 0\}$ a vector space with the operations:
ADDITION: $u \oplus v = (xa, yb, zc)$ where $u = (x, y, z), \quad v = (a, b, c) \quad \in$ V
MULTIPLICATION BY A SCALAR: $\lambda \otimes$ u $= (x^{\lambda}, y^{\lambda}, z^{\lambda})$ where u $= (x, y, z) \in$ V and $\lambda \in$ R.
Let W be the subspace of all points in V that lie on the plane $z = 1$.

1. Write down two vectors of W.
2. What is the null vector of W?
3. If $v = (3, 2, 1) \in W$, what is $-v$?
4. Are the vectors $(2, 2, 1)$ and $(1/2, 1/2, 1)$ of W linearly independent?
5. Is the set {(3,3,1), (1/3,3,1)} a basis for W?

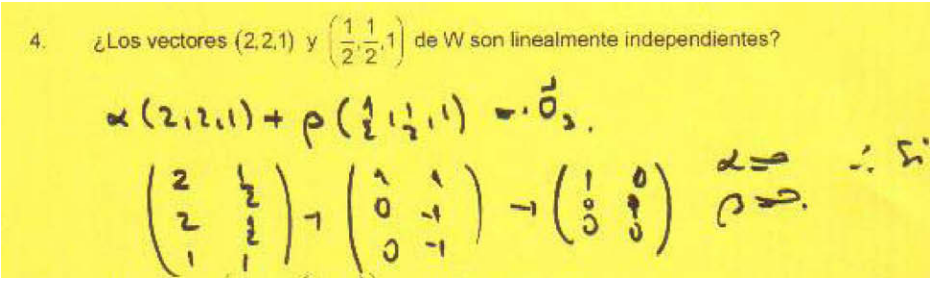
When trying to solve part (2) of this question S7 first thinks that $(0, 0, 0)$ would be the null vector. But when he realizes that it does not belong to W, he changes his answer to $(0, 0, 1)$. He writes the following to justify his response:



We observe in what he wrote that he makes a mistake as to which vector should be obtained when the null vector is added to any vector. Afterwards as he tries to find the vector $-v$ where v $= (3, 2, 1)$, the following conversation takes place:

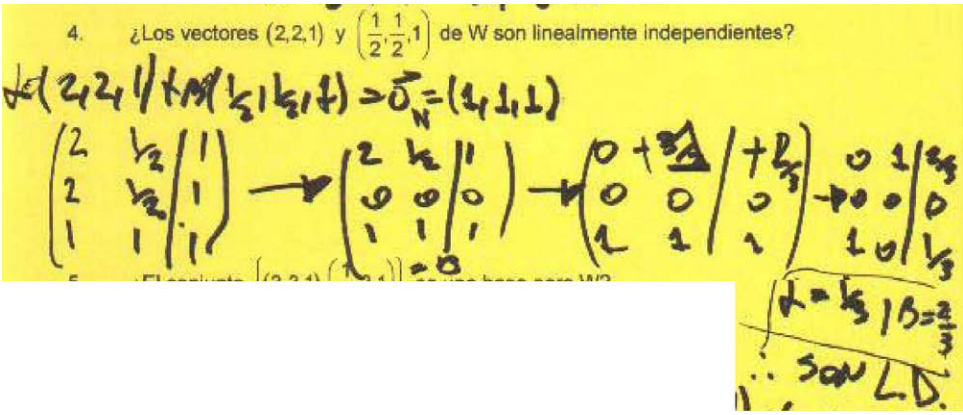
S7: It has to be $(-3, -2, 1)$ so that it is in the subspace, so that if I add it with the other one, ahhhhhh no, no, it cannot be this one.
I: Why not?
S7: Because if I operate, it is supposed to come back to the null vector but here it won't happen, with this operation. It has to be coherent with that, but. . .

S7 cannot resolve this situation. In part (4) of the question, in order to find out whether the given vectors are linearly independent, he puts them into a matrix and performs row operations without paying any attention to how the operations are defined:



This student has a weak binary operation schema that leads him to confusion when the operations defined on the vector space are not the usual ones. Because of this he cannot interpret the linear independence question appropriately, either. Relations between the notions of linear independence, null vector, vector space axioms and binary operations are not constructed completely.

Another student S1 answers the first three parts of the question correctly. However to answer the 4th and the 5th parts, he uses the usual operations instead of the given ones, although he equates the equations to $(1, 1, 1)$ which is correct. Below we show how he solves the 4th part:



However when the interviewer is about to pass to the next question, he seems to doubt his answers and says:

S1: I was just thinking that. . .if it is linearly dependent they should be able to be written as a linear combination of one in terms of the other.

. . .

S1: What happens is that, OK, for example here, this multiplication, what is it really? It's the circle operation. What does it do?

I: Let's see.

S1: It raises to a power, that is in reality I am solving an equation, let's see. For example I'll check if what I came up with is right. 1/3 yes. . . but I'm getting a square root. . . I get this to the power 1/3, this to the power 1/3, one, plus this raised to the power 2/3, all this gives me 1/4 and the sum is defined as a scalar multiplication, scalar, that is 1/3, that is, let's see, 2 raised to the power 1/3 times 1/2 raised to the power 2/3. If this is cubic root of 2 times cubic root of. . . that's it, 1/2, sorry, times cubic root of 1/4 and, that's it: cubic root of 1/2 is different from 1.

I: What happened?

S1: It doesn't work. . . It should give me 1, at least the first coordinate. So, what I'm saying is that, summing up the third part (he means the fourth part), what did I say?

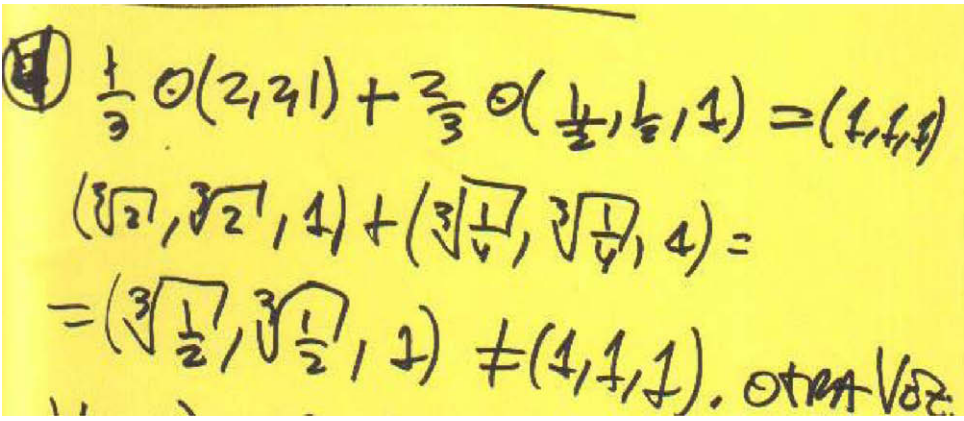
. . .

S1: So I have to do it another way.

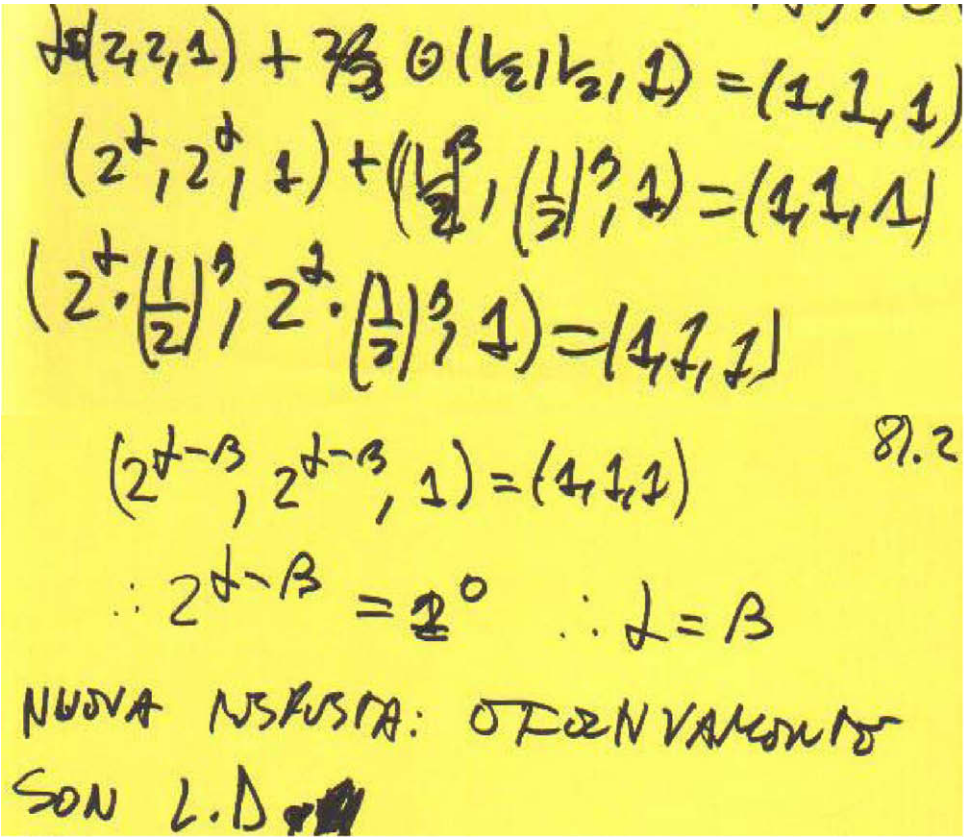
E: What happened?

S1: Actually it happened to me again that I didn't take into account how they were defining the operation.

Below we reproduce S1's work:



He proceeds to solve part (4) again, this time he does it correctly:



At the end he says:

S1: They also are linearly dependent, but the alpha and the beta are not necessarily those (referring to the values he had calculated previously).

The difference between S1 and S7 is that in S1's mind the concepts of linear independence, vector space and binary operations are related to each other. Although at first he does not take into account the given operations, the fact that he has constructed strong schemas of binary operation and vector space allows him to realize his error and the causes for it.

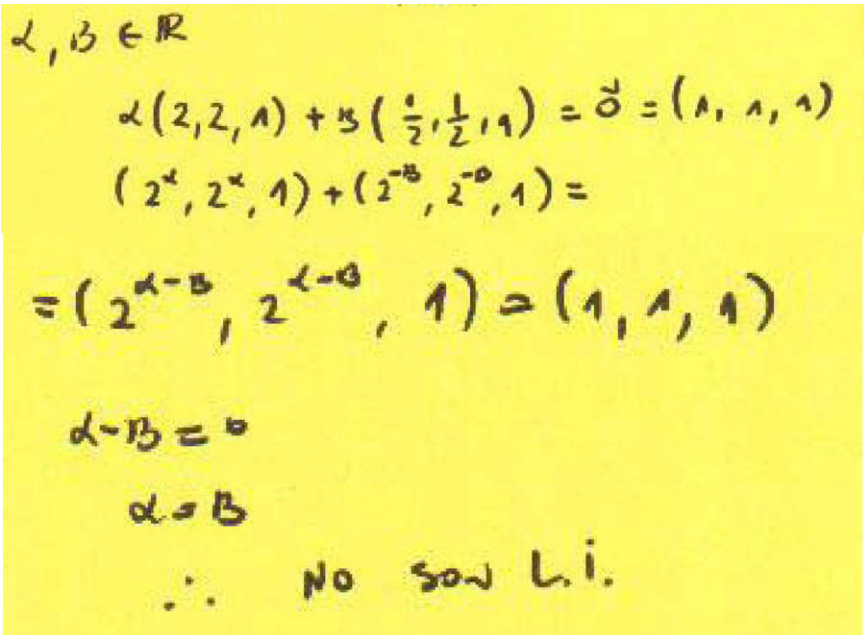
Now let's see how another student S2 explains why $(1, 1, 1)$ is the null vector:

S2: We know that the null vector, when added to another vector, has to give the same vector. So it has to be $(1, 1, 1)$. And I also thought that when α is zero, well, actually this is what gave me the vector, when α is zero it has to give me the zero vector, and here we have x to the power zero. . .

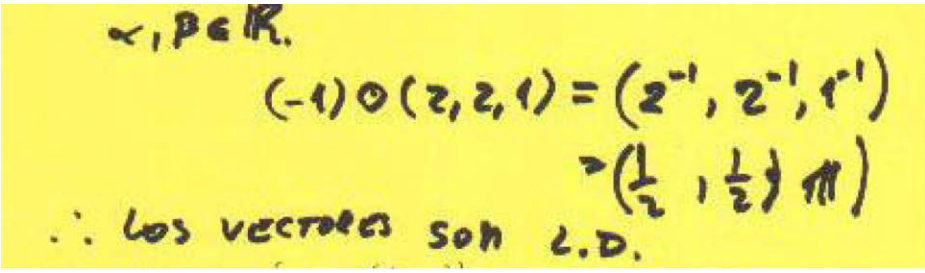
He also explains how he found $-v$ in the third part of the question:

S2: What is $-v$, that is the inverse? Then, eee, what do we have to see? That $-v$ will be a vector that belongs to W which when added to this vector has to give the neutral vector, $(1, 1, 1)$.

For the fourth part, S2 correctly uses the operations and the zero vector to check for linear independence:



On the other hand S3, to answer the fourth part uses the following strategy:



These two students, S2 and S3, show that they have constructed strong vector space and binary operation schemas, which, in turn, are related to the concept of linear independence adequately.

7. Discussion

In this paper we proposed a genetic decomposition which predicts how students might construct the vector space concept as a schema. Our empirical evidence in this article focused on the parts of the proposed genetic decomposition that have to do with the coordination of the processes of the two binary operations, and the relation of the vector space schema to other concepts. Through the questions that we designed and applied in the form of a questionnaire and an interview, we observed that when students lack the prerequisite constructions, it becomes very difficult for them to develop a sufficiently strong schema of the vector space concept. We also conclude that the coordination between the operations vector space structure is not a trivial one cognitively, although such might be the case mathematically.

Related to these research results our first pedagogical suggestion is that there should be special emphasis put on the construction of the binary operation schema, giving students the opportunity

to experiment with different kinds of sets and binary operations, so that they develop flexibility in thinking about structures other than the ones containing the usual operations. Our next suggestion has to do with the way the two vector space operations are related to each other. Usually in teaching the vector space concept, the axioms that combine the two operations are treated like any other axiom, whereas cognitively there is a demand that involves the coordination of two processes. Again, there should be activities designed to facilitate such a coordination. The questions that we reported and students' answers to them can be used in designing instructional strategies to favor the necessary connections that form part of our genetic decomposition.

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