Conditions for promoting reasoning in problem solving: Insights from a longitudinal study

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Abstract

This paper describes insights on how to promote mathematical reasoning in problem solving based on the mathematical experiences of participants in a long-term study in which the students engaged in strands of well-defined, open-ended mathematical investigations, as a context for research on the development of particular concepts and ways of reasoning. Over the years, the students demonstrated ways of working in which sense making became a cultural norm and collective and individual sharing and justifying of ideas was a common practice. The paper examines the environment that enhanced the development of these and other qualities. The insights address aspects of task design and researcher role in the students’ mathematical activity. Both were central in enhancing the students’ engagement in thoughtful and meaningful problem-solving activity. The paper addresses the relationship between problem solving and mathematical reasoning from the perspective of the long-term study and provides an examination of students’ problem solving over time, both from a behavioral and introspective perspective.

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1. Background of the longitudinal study

The purpose of this paper is to provide insights on how to promote mathematical reasoning in problem solving based on the mathematical behaviors and reflections of a group of students who participated in a long-term research study. As a context for research on the development of mathematical ideas and ways of reasoning, the participants engaged in mathematical investigations throughout their public school and early university years. The investigations focused on areas of combinatorics, probability, algebra and calculus from a problem-solving perspective. We report results that derive from an extensive body of experience from both longitudinal and cross-sectional studies at Rutgers University and elsewhere on how mathematical meaning is built by learners. The research is based on an extensive collection of videotapes and related work from three New Jersey school districts: working class, urban and suburban as well as elsewhere.

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Over the years, the students demonstrated advanced mathematical reasoning and ways of working that emphasized sense making and collective and individual sharing and justifying of ideas. The researchers developed a repertoire of tasks and research interventions to facilitate communities of learners and practitioners as they engaged in major changes, not only in how they worked in mathematics but also in how they thought about, understood and made choices about what they did. We examine the students’ mathematical experience in the longitudinal study and describe conditions that proved useful in enhancing the students’ thoughtful problem-solving activity. We rely particularly on video data on the students’ mathematical activity in the longitudinal study, the students’ reflections on such experiences and published studies in connection with the longitudinal study.

Finally, the present study addresses a documented need for more research on the relation between problem solving and mathematical reasoning and an examination of problem solving over time, both from a behavioral and introspective perspective. A comprehensive description of the goals, activities and history of the longitudinal study is available elsewhere (Francisco, 2004, 2005; Kiczek, 2000; Maher, 2002, 2005; Maher & Martino, 1996a, 1996b, 1989; Powell, 2003).

2. Theoretical framework

This paper addresses three issues regarding research on the relation between problem solving and mathematical reasoning. Studies have shown that reliance on word problems that can be unambiguously solved through computational proficiency and obvious arithmetic operations hinder realistic mathematical reasoning, whereas emphasis on mathematical modeling and interpretive skills enhances it (Greer, 1997; Verschaffel, 1997). However, more research is needed that can help provide a “coherent explanatory frame” as to how problem solving and mathematical thinking fit together (Schoenfeld, 1992). This study makes a contribution on the issue by sharing a number of insights on how to promote sense making and reasoning in problem solving.

Research has also shown that beliefs play an important role in problem solving (Kitchener, 1983; Schoenfeld, 1983). However, few studies have systematically examined problem solving from the interplay of the students’ behavior and their mathematical beliefs. The predominant approach has been to consider problem solving from either the perspective of students’ mathematical behavior or their beliefs and rarely from both perspectives. As a result, the study of problem solving has focused mostly on generating and describing taxonomies of students’ problem-solving heuristics in as much detail as possible so that they can be useable by other students. This is how Schoenfeld (1987) once described the cognitive science approach to the study of problem-solving heuristics, which prompted Balacheff and Gaudin (2001) to ask whether there is a level of description of the heuristics which guarantees a reliable transfer from one operator to another and provides an accurate indicator of the student’s mathematical understanding. This study examines the problem-solving activity of a group of students from the perspective of their mathematical behavior and their views on it. It also builds on the longitudinal research to address the need documented elsewhere (Schoenfeld, 1992) for an examination of problem solving over time.

Problem solving has been associated with different meanings, which reflect different views about mathematics and mathematical learning. For example, Schoenfeld (1992) distinguishes three traditionally different views of problem solving. In one, problem solving is the act of solving problems as a means to facilitate the achievement of other curricular goals such as teaching mathematics, motivation, recreation, developing and practicing mathematics skills. In another, problem solving is a goal, out of many, in itself of the instructional process. It is a skill or piece of knowledge that is worth teaching in its own right. Finally, when challenging problems are involved, problem solving can be viewed as a form of art, as what mathematics is ultimately all about. Our perspective of problem solving recognizes the power of children’s construction of their own personal knowledge under research conditions that emphasize minimal intervention in the students’ mathematical activity and an invitation to students to explore patterns, make conjectures, test hypotheses, reflect on extensions and applications of learned concepts, explain and justify their reasoning and work collaboratively. Such a view regards mathematical learning and reasoning as integral parts of the process of problem solving. This means that the results of this study can also be understood as conditions for promoting meaningful mathematical learning.

3. Method

The longitudinal/cross-sectional study took place in three New Jersey school districts: Kenilworth (grades 1–12; college; post-college), New Brunswick (grades 4–8; summer, grade 11) and Colts Neck (grades 4–7). Over the course
of the study, approximately 80 students from these districts were videotaped doing mathematics. Depending on the district and the mobility of the students, we followed individuals for different lengths of time, varying from 3 to 18 years.

The database of the study included an archive of approximately 3500 videotapes (CDs) of problem-solving sessions and individual and small-group interviews; session descriptions with accompanying observer/researcher notes; student written work; video transcripts; and recent follow-up questionnaire results. We report on the work of a group of students from the Kenilworth study who participated in the long-term research since its origin in grade 1 (1989) to their university years. Our insights are derived from an examination of (1) the students’ mathematical behavior and written work in problem-solving activities, (2) the students’ reflections on their longitudinal study experiences collected through clinical interviews (in grades 11 and 12) and follow-up questionnaires (during their college years) and (3) results of detailed analyses of the development of student ideas and forms of reasoning as reported in both published and unpublished research reports. In our presentation, we refer to the students’ observed mathematical behavior and written verbal reflections on their activity. We also indicate results from relevant studies published over the years in connection with the research. The videotapes of the student mathematical behavior were selected from the database of the Kenilworth longitudinal study, particularly the students’ mathematical activity on three problems and variations and extensions of them: Tower Problems, Pizza Problems,2 and the World Series Problem.3 Many research sessions in the combinatorics and probability strand were based on these and versions of these tasks.4 The students often referred to them in their reflections.

The gathering and analysis of the interview and questionnaire data followed a phenomenological approach, which allowed the students to choose aspects of their experiences that were relevant to them and how they wanted to articulate them to us (Creswell, 1998; Giorgi, 1985; Moustakas, 1994). The analysis of the problem-solving data used a process suited for the study of mathematical reasoning based on videotape data (Powell, Francisco, & Maher, 2001, 2003). The approach relies on the identification, description, coding and interpretation of critical events or episodes. In this study, the critical episodes referred to the students’ different forms of mathematical reasoning and the research conditions associated with them. Issues regarding the use of verbal data are discussed in more detail elsewhere (Ericsson & Simon, 1984). In this particular study, behavioral and verbal data were initially analyzed independently of each other. Later, cross-data analysis was carried out and emphasized the search for a potential fit or consistency between results from both kinds of data. As suggested elsewhere (Erikson, 1992), a search was also conducted for instances in the database that might contradict particular claims before accepting them as findings. Finally, the use of research reports as sources of inference helped provide a level of description of findings beyond the scope of the present study.

4. Results

Sections 4.1–4.6 discuss the contributions of this study on how to promote mathematical reasoning in problem solving based on the longitudinal study.
4.1. Basic ideas

Mathematics is often construed as a system of complex relationships involving mathematical concepts. Mathematical reasoning is also associated with the ability to discern and articulate such relationships. However, the longitudinal study experience suggested that these are not the only sources of forms of mathematical reasoning. Basic mathematical concepts can also be cognitively challenging. For example, the World Series Problem showed how cognitively challenging the concept of a sample space could be to students. The students worked on the problem for approximately two hours over three days. They considered a number of strategies before deciding to solve the problem by using the probability ratio. Eventually, they came up with the answers, $P(4) = 2/16$, $P(5) = 16/32$, $P(6) = 20/64$ and $P(7) = 40/128$, whereby they determined the numerator by listing all possible winning game combinations and computed the denominators as powers of two (for example, the denominator of $P(n)$ was computed as $2^n$, for $n = 4, 5, 6$ and $7$). However, the analysis of the students’ problem-solving work revealed that the students’ major challenge was determining the denominators of the probabilities. For example, as they worked on computing $P(4)$, Romina suggested that the denominator could be 7, the maximum number of games that the teams would play before a winner of the series emerged. Mike agreed with Romina, whereas Ankur and Jeff disagreed, as follows from the following excerpt:

JEFF: All right. So after the four, for winning in four games –

ROMINA: Should it be over seven, though?

MIKE: It should be over – over seven, ’cause it’s four out of seven games.

ANKUR: But this one wouldn’t be over seven [referring to $P(4)$].

JEFF: It wouldn’t be.

ANKUR: It wouldn’t. None of this would be over seven.

Ankur suggested that computing the denominator as a power of two. However, as the students worked on computing $P(5)$, the issue about the denominator came up again. Romina asked if the denominator for $P(5)$ would be $2^5$ or $2^7$:

ROMINA: Wouldn’t it be two to the fifth? Or would it be two to the seventh?

JEFF: Two to the seventh is a hundred twenty-eight. And, but, like B, B, B, B, B – like B seven times wouldn’t count. But on the other hand, the sum –

ROMINA: But, I’m saying for this one, it’s eight over – would it be eight over two to the seventh or two to the fifth? Maybe to the fifth? Well because –

The students agreed to compute the denominator of $P(5)$ as $2^5$ and used a similar strategy to compute the denominators of $P(6)$ and $P(7)$. However, the issue about the sample space continued to come up in a different way. For example, when the students found it time consuming and challenging to list all winning game combinations for the numerator of $P(7)$, Jeff suggested computing it by subtracting from 128 [sample space for $P(7)$ computed as $2^7$] the numbers 2, 8 and 20 [winning game combinations for series ending in 4, 5 and 6, respectively]. The students did not notice that these numbers came from different samples spaces:

JEFF: You have twenty-eight here, so you’d subtract two because of the four games. There’s two that would cancel out, like –

ANKUR: What do you mean?

JEFF: And then –

ANKUR: You’d subtract eight.

JEFF: Eight. Ten – no, then eight, twenty –

ANKUR: Twenty.

BRIAN: Those aren’t the best [inaudible].

JEFF: Thirteen.

BRIAN: They can’t be factors in the seventh game.

ANKUR: That’s what I’m thinking.

JEFF: Well that’s why they’re not – that’s why you subtract them.

ANKUR: That’s why you subtract them. Yeah.
JEFF: But then you get like a number like ninety-six.
BRIAN: Out of what? One twenty-eight?
ANKUR: Out of what?
JEFF: I’m not sure.

Finally, in a subsequent problem-solving session, the students were asked to compare their solution for the World Series Problem to a different one proposed by a group of graduate students. The proposal indicated the probabilities as follows: $P(4) = \frac{2}{70}, P(5) = \frac{16}{70}, P(6) = \frac{20}{70}$ and $P(7) = \frac{40}{70}$. Confronted with this option, the students were visibly unsettled by the different “solution” even though they did not relinquish their solution. They responded by engaging in interesting discussions about the differences in the denominators. The eventually explained their decision to compute the denominators of the probability ratio as powers of two by analogy or isomorphism with the Tower Problem $n$-cubes tall and selecting from two colors, which they had previously solved. However, it should be noted, that the issue of correctness was not settled immediately. Mike, for example, pursued thinking about the problem for a few months before he indicated that he was convinced of the validity of the groups’ original solution and the incorrectness of the alternative one. He eventually pointed out the flaw in the graduate students’ proposal after solving another problem, the Problem of Points.

In summary, a basic concept, such as the notion of a sample space, can be challenging to students and promote complex forms of reasoning. The students exhibited disequilibrium when being confronted with a challenge to their ideas. However, they also eventually mapped the World Series problem to the Tower problem to explain their solution. Mike’s case, in particular, shows that persistence and willingness to discuss further the reasonableness of the original ideas makes possible the building of deeper understanding and more elaborate schemes.

4.2. Complexity and simplicity

Task design is crucial for sustained engagement of students in problem solving and for promoting sense making and mathematical reasoning. Traditionally, complex or “difficult” mathematics problems are partitioned into simple or “easy” parts, which are then presented to students in ascending order of cognitive difficulty, as a “build-up” to the students’ understanding of the complex task. This approach is consistent with the epistemological view that students build increasingly more complex knowledge, in atomistic fashion, over time. The approach depends on the students’ ability to “put things together” at the end. However, the longitudinal study provided evidence that this may not take place. Also, when the students do manage to relate the parts together at the end, the resulting knowledge is often not necessarily personally meaningful to the students. The longitudinal study emphasized the advantages of a different approach, which proved useful in promoting meaningful and thoughtful mathematical activity. This approach involves presenting, first, the complex task to the students, as opposed to scaffolding the “easy” parts taken from the task. For example, regarding the Tower Problem, third grade students were asked to build 4-tall towers straight away; fourth grade students were asked to build 5-tall towers and higher. With the Pizza Problem, the students were first asked to find the number of pizzas that could be made when selecting from two, then four toppings that permitted placing a topping on half a pizza. The analysis of the students’ work on the problems revealed that these were not easy or simple tasks for the students at their grade level. However, the students were able to come up with interesting strategies and ways of thinking about the problems. For example, some students organized the towers by cases; other students used an inductive argument that showed how the towers “grew”; others provided evidence of reasoning by contradiction to support the claim that they had accounted for all possible towers within a given category; some used recursive reasoning to account for all possible towers for a certain case (Maher & Martino, 1996a, 1996b). An interesting way used by some students to recursively represent their solution-involved notation. The use of 0s and 1s to represent the presence of a topping and, later, binary numbers to represent particular topping choices, triggered recognition of

5 For a comprehensive discussion on the World Series Problem, see Kiczek (2000).
6 The Problem of Points: Pascal and Fermat are sitting in a café in Paris and decide to play a game of flipping a coin. If the coin comes up heads, Fermat wins; if it comes up tails, Pascal wins. They start playing, and Fermat wins. But then a strange thing happens. Fermat is winning, 8 points to 7, when he receives an urgent message that his child is sick and he must rush to his home in Toulouse. The carriage man who delivered the message offers to take him, but only if they leave immediately. Of course, Pascal understands, but later, in correspondence, the problem arises: how should the 100 Francs be divided.
the likeness of pizza and tower problems under certain condition. Also, the “simpler” Pizza Problem that involved
the placing of toppings on whole pizzas later became trivial for the students once they solved the Pizza with Halves
Problem.

In terms of task design, the researcher may need to consider carefully what counts as a difficult or complex task.
This, of course, will depend on the particular students involved and their earlier experience working with these and
with similar problems over time. However, our experience shows that students are more likely to build durable and
meaningful understandings of the mathematics by unveiling, for themselves, the complexity of the task. Eventually,
the students, in a natural way, identify the components and propose interesting variations and extensions.

4.3. Strands

The longitudinal-study experience highlighted the advantages of a task design model based on a strand of problems. A
strand is a series of related tasks designed around identified mathematical concepts with comparable levels of difficulty
and similar problem-solving structure. For example, the Tower Problems, Pizza problems and the World Series Problem
were part of the combinatorics and probability strand. Some of the problems in the strand were extensions of previous
problems and helped test the durability and depth of the students’ understanding of the underlying concepts in the
strand. The major advantage of the strand approach in the longitudinal study is that it enhanced the student’s ability to
overcome cognitive obstacles in problem solving. Usually, when students encounter a cognitive obstacle in problem
solving, they are told to keep on working on the problem until there is a breakthrough. However, the strand approach
seeks to help students overcome the obstacles by other ways than just time and effort, no matter how important these
may be in problem solving. More specifically, in the strand approach the researcher allows the students who have
worked long enough on a task to move on and engage in another related investigation. This provides an opportunity
for students to revisit the same issues or concepts in a different context, which may be more cognitively appealing
or familiar and increase the potential for a breakthrough. Often, the students manage to overcome the difficulties or
obstacles by mapping different problem situations and solve one of them by some form of isomorphism. For example,
the excerpts below provide a verbal account of the students’ mapping of the World Series Problem to the Towers
Problem, which helped them compute the denominator of the probability ratio [sample space] as powers of two, or
equivalently, the number of towers that could be built when choosing from two colors of height equal to four, five, six
or seven, when computing \( P(4), P(5), P(6) \) or \( P(7) \), respectively:

ROMINA: How many possibilities [winning and all possibilities]?
ANKUR: There can only be two ways [series ending in four games]
ROMINA: Isn’t it – yeah, two n? [For the total number of possibilities]
JEFF: Yeah. All right, so say it’s two to the seventh.
ROMINA: How much is that? I don’t know.
ANKUR: For this, you’ve gotta find all possibilities with –
ROMINA: Should it be over seven, though?
ANKUR: Eight of these. It’d be over, like, total possibilities of –
JEFF: Yeah, the total possibilities is eight, right?
ANKUR: They have eight ways of winning but it’d be over –
JEFF: Oh. Eight over one – no – well, how do we find out?
ANKUR: I’d be over – the total possibilities of two, like two – two colors and five things [towers 5-tall with two
colors].

It is interesting that, over time, the students spontaneously referred to the same particular set of problem(s) to
solve other problems through mappings. This suggests another advantage of a strand design that problems could be
used by students in ways similar to the use of prototypes as a means for building meaning in mathematics (Dörfler,
2000). In particular, the Tower and Pizza problems were the most frequently used prototypical problems by the
students.

Finally, in the following excerpt, Romina acknowledges the importance of the prototypical use of problems in
learning. She claims that she made use of some of the prototypes from the longitudinal study in her college mathematics
learning:
During my second semester of calculus in college, we concentrated in probability. My friends and I came together every week to do our calc homework and study for exams. Several of the members of this group were very strong in math, particularly when it came to the typical “crunching” of numbers. For some reason, they seemed to have some problems with the probability, mostly because it was more conceptual than just a quick formula. I was able to help them by bringing out one of the Rutgers problems, good old towers. By starting with a basic idea – towers – we were all able to build on the concepts and understand the basics of probability.

It is not surprising that Romina refers to prototypes as basic ideas. For her, they were the building blocks for more complex forms of reasoning.

4.4. Centrality of ownership

This is an issue that is not often associated with problem solving, perhaps because it is not so much about actions taken in problem solving, as it is about the students’ attitude. However, the students’ ownership of their mathematical activity was central in promoting students’ successful problem solving in the longitudinal study. Ownership of mathematical activity means that the students’ mathematical ideas, representations, justifications and decisions are emphasized over those of teachers, researchers or other experts. Ownership was the issue most discussed by the students in their reflections. For example, Mike suggested that it was a distinctive aspect of the longitudinal study:

The Rutgers math [sic longitudinal study] felt different from the math we did in class because, when we would do the Rutgers sessions, it was like the kids were running the class. You would ask us something and then step down and let us do our thing. When I was in [inaudible] the teacher would just like mainly talk and you’d write things down. Students don’t have that much input in what they were learning.

In the following excerpt, Mike also suggests that ownership helped the students build durable forms of mathematical understanding:

I think the point of having kids run the class is that, if you tell a kid something, they might understand; they might not; but if a kid says it himself, then obviously, he understands it. You get to understand things a lot better if you’re running the class. Obviously, the teacher understands it, but who knows if the kids do? They could just be coping it down. A week later, they’ll never remember what they did.

Romina indicates the importance of explaining and knowing things in her own way so as to develop personally meaningful knowledge:

They might throw out, “Oh, do you know this rule, and guy?” I’m like, “No, but if you sit me down, maybe I know it, “I know it in my own way, not in their way. Everything I explain is in my own words, not in anyone else’s words. It’s not from some mathematician from a thousand years ago, because I don’t know that. I didn’t know what the pyramid [Pascal’s Triangle] was called. I just know everything in my own way. Everything has Romina’s definition to it.

However, the following excerpt from Romina also recognizes developing a sense of ownership may take time. She looks back at her problem solving from earlier years. At first, it was new and unfamiliar. However, she also notices that the acceptance of ownership evolved over time and with this, came more independence in their problem solving:

In fourth grade, I didn’t know who you [researchers] were. Now we’re comfortable with you. You’ve been our teachers for ten years. That’s what you’ve been to us, so now it’s easier, and we know what’s expected of us, what we have to do. Before we would wait for you to give us a little start or a little push and point us in the direction. Now you hand us a problem and you just kind of leave, and we just do it ourselves. We just start experimenting and see what we can give you.

From Romina’s perspective, learning to engage in doing independent problem solving with other students was a journey that became increasingly more comfortable over time. It is interesting that she refers to the researchers as her teachers, and recognizes their decreasing presence in presenting problems. She acknowledges their acceptance of responsibility to “just do it ourselves.” A sense of ownership of the problem solving was accompanied by an increase
in confidence and engagement. Finally, the students developed ownership of the process of doing mathematics and of the resulting forms of the knowledge that they built together.

4.5. Justifying versus proving

Proofs are paramount in mathematics and mathematical learning. They provide legitimacy to mathematical solutions, claims and forms of reasoning. An important and widely published finding in the research literature of the longitudinal study is that children, in a natural way, learn the idea of proof (Maher, 2002, 2005; Maher & Martino, 1996a, 1996b, 1997, 1998, 2001). It is important to be aware of the complexity of the students’ arguments and the conditions that were in place when the students promoted meaningful proofs. Often proofs are associated with strong mathematical rigor and formality, which makes learning how to prove a difficult subject for students. As a result, proofs are usually introduced later in school, sometimes, in high school and usually in undergraduate college where special courses are designed to help students learn to write valid formal proofs.

Another important finding from the longitudinal study regarding proofs is that students’ can successfully engagement in proof making, even at a much earlier age, is enhanced when the focus is on building convincing justifications for their mathematical claims and not so much on the formal writing of them. As such, proving becomes an integral, not separate, part of the problem-solving process and promotes the building of personally meaningful arguments and ways of articulating them, in contrast to attempting to fit ideas into a predetermined way of writing proofs. For researchers, the main question is, “Why?” or “How can you convince someone?” as they try to encourage students to come up with convincing arguments. For example, when the students worked on the Tower Problems, they came up with interesting ways of justifying their answers. Third grader, Meredith, organized blocks of 4-tall towers, selecting from two colors, by cases to argue that all possible towers, were built (Martino & Maher, 1999). Fourth grader, Stephanie, used symbols R and B to represent red and blue towers, respectively, to show that all possible 3-tall towers were built. Then, she generalized her solution for towers 10-tall, using a doubling rule (Maher & Martino, 1996b). Fourth grader, Milin, built towers 5-tall and illustrated the doubling rule using an inductive argument (Alston & Maher, 1993). He later extended his solution for towers n-tall (Maher & Martino, 1997). Fourth grader, Brandon, used a case argument to show all possible 4-tall towers, selecting from two colors by recognizing the equivalent structure of this problem with building pizzas, selecting from four toppings (Martino & Maher, 1999). The students did so by focusing on making sense of the problem for themselves individually and as a group (Martino & Maher, 1993).

The longitudinal study experience also suggested that students might need time to develop and come to understand the importance of a way of working based on sense making and the justification of ideas, as implicit in the following excerpt by Romina regarding the early attempts by the researchers to introduce the justification of answers in the longitudinal study:

My first memories of Rutgers [sic longitudinal study] were, I got pulled out of class one day, and I didn’t know why, and I got put into a special class. Which is kind of scary, because you don’t want to be different back then. I remember you came in and we all sat there and we had these cameras, and you just gave us things to play with, and it wasn’t that bad—[laughter]—everyone kept asking why, and we never understood that until about this year. It’s hard to believe. We were scared, but then you made it fine. It wasn’t that bad.

Finally, Jeff also acknowledges the importance of rigor in mathematical proofs and, in particular, of sense making in building proofs:

Just giving an answer was not enough. You had to have a good structural record. It’s almost like doing a proof. You need to show every step from point A to point B; you couldn’t just skip some things and jump around. You had to go straight, and everything had to be written out, and good [sic well], and understanding, and if you had a problem with somebody, to ask another questions about it.

It is interesting that Jeff labels the justifications in the longitudinal study, as “almost like doing a proof,” distinguishing them from the common view of proof which emphasizes rigor and formality.

4.6. Collaboration

There is substantial agreement that collaborative work among students can be beneficial to students’ mathematical learning. The predominant view of collaborative work, however, is of students helping each other overcome a cognitive
obstacle by providing a missing piece of knowledge. The success of the collaboration depends on the extent of knowledge base of the participants. When group members cannot provide the missing piece of knowledge, collaboration fails. The longitudinal study experience highlighted the importance of another form of collaborative work among students, whereby the group members rely on each other to generate, challenge, refine and, accordingly, drop or pursue new ideas. The students do not rely so much on their accumulated knowledge as much as they do on their ability to think together and build an understanding of the problem situation involved. In particular, they discursively come up with convincing arguments for their answers when solving problems (Powell, 2003; Uptegrove, 2005). For example, the following excerpt, taken from the students’ work as a group on the World Series Problem, illustrates how the students collectively built their problem-solving strategy by proposing and critically evaluating their ideas in the group:

ROMINA: They can go all seven or they could go all four. So, it would be A, A, A and B, B, B – Team A and Team B? So, in four games, would it be, like, one-half of a chance? Or would we have to write it out with – using all seven?
JEFF: See, I think that it’s the hardest to win it in four games.
JEFF: Definitely the hardest.
ROMINA: Yeah, exactly.
JEFF: So, it wouldn’t be one-half.
BRIAN: Isn’t it the odds – the odds of winning one game, times the odds of winning one game, times the odds of winning one game?
ANKUR: Look, it’s a fifty percent chance of winning the first game.
BRIAN: So, it’s like, half times a half – no, wait – remember the odds get harder to win two in a row, like a coin flip.
ROMINA: Yeah, that’s how you do it: A half times a half times a half times a half.
JEFF: If that’s the case, what is it [the probability for five]?
ROMINA: Is it one thirty-two?
BRIAN: Two times two is four, times two is eight. Three times three–
ROMINA: Oh, never mind, I get it. Now, would you have, for five games, like, would it be like that \[\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\]?
ANKUR: Hopefully, the odds of winning are [getting bigger]–
JEFF: We’re never going to get – it’s never going to equal up to one, though.
BRIAN: Does it have to?
ROMINA: No, but, I was thinking, you know how we do, like, uh, like A –
ROMINA: [Looks at her paper] Wouldn’t you have easier odds of winning in six games than in four?
JEFF: Yeah.
ROMINA: Doesn’t it get less, though?
JEFF: That’s why it’s wrong.
ROMINA: Okay [crosses out what she has written].

Romina starts by suggesting that the probability of the series ending [being won] in four games is a half since there are only two possible winning cases [AAAA and BBBB for Team A and Team B, respectively]. Jeff argues that “It’s hardest to win it [the series] in four games” and Romina’s idea is rejected. Brian suggests multiplying consecutively the “odds” (sic probability) of a team winning a game. The idea is rejected after the students notice that the chances of either team winning the series do not increase with the number of games played. As previously discussed in this paper, the students eventually solved the problem by using the notion of probability as a ratio and listing winning game combinations. In particular, the students worked together to control for variables when listing winning game combinations:

ROMINA: I’m just so lost. Like, I completely messed this up [her listing]. I – do you want me to go through some of mine to see if we have them? You did the same. You went in the same pattern I did.
JEFF: Do you see what I’m saying with that?
ROMINA: [Looking at Jeff’s work] You did. Then you moved the B over and then you moved the other B over.
ANKUR: Brian, did you get eighteen for six?
BRIAN: Forty-two for game seven.
ANKUR: Are you sure none of them are like – I got twenty.
JEFF: I think it’s a high number that is six –
BRIAN: It’s getting blurry after a while.
BRIAN: Maybe there’s one I forgot. All I got to do is find one, and then the opposite for it.
ANKUR: Yeah, I know what you mean.
ANKUR: [To Brian] I got the two that you missed.
BRIAN: So, you got twenty for that now, so if it’s all right –
ROMINA: Ankur, what are you doing down there?
ANKUR: Just finding the two that Brian missed. I found them.
ROMINA: How did you get twenty? Did you write them all out?
ANKUR: I wrote them out.

The excerpt suggests that the students collectively built their solution by sharing, comparing and critically evaluating their ideas and strategies, as they come up with them. They thought together as a group through the problem and built a solution with input from all of them.

The students acknowledged the importance of group work in their reflections. One student argued that it “instilled in me an attitude towards learning that is self-motivating and independent from a teacher and at the same time co-dependent with other peers.” Another student, after graduation from college and now working as a consultant, described in detail the group work in the longitudinal study and mentioned its advantages, which included personal and mathematical growth:

The Rutgers math program has been of value to me and to the rest of the group in more ways than just math. The program did teach us how to think about math, but more so developed a style of learning among our group. It is difficult to explain how the process took place. When we were first approached with a problem, the room became quiet. We all read through the details and let it all register. Then, we started to ask each other questions and work through our thoughts as a group. We were cooperative even when we argued; it resulted in a positive because we uncovered a layer of the problem that wasn’t as clear before. Working through every step of the problem helped us understand it better and organize our thoughts as if we were recreating the entire problem from the very beginning. This process came into play later on when we had to present our findings. Being in front of a group of professors can be intimidating, but we became so accustomed to the process, that it was almost natural and we expected to be asked difficult questions that would test our thought process. We approached every problem knowing that we would have to face these questions, and as a result, we were thorough, organized and ready to win over the audience. Most of us have carried this style over to other parts of our lives. We became comfortable in expressing our well thought out solutions and ideas. We become accustomed to working through problems with other people, listening to them and coming to a solution together. To this day, we all value our analytical and communication skills that we developed through the program.

The excerpt acknowledges that developing an efficient ways of working together is likely to take time. Finally, Jeff related group work with the building of proofs:

We didn’t know if we were right or wrong. You only knew so much, but I would have my idea about how to get to a certain point and you might have the same idea about how to get to it. But getting there was the hardest part. That is what we were arguing about, the right way to get there, the right way to make sure we covered the basis, how to make sure, how to prove what we needed to accomplish.

Implicit in the excerpt is the idea that proof making is discursive activity.

5. Conclusions

We set out to contribute insights on how to enhance mathematical reasoning in problem solving based on our experience in a longitudinal study in which a group of students engaged in well-defined, open-ended mathematical investigations, as a context for the development of particular mathematical concepts and ways of reasoning. The sources of inference include the students’ mathematical activity, their reflections on their experiences in the longitudinal study and research reports related to the longitudinal study. The paper addresses the relationship between problem solving
and mathematical reasoning, from the perspective of the long-term study and provides an examination of students’ reasoning over time, both from a behavioral and introspective perspective.

The results of the study highlight the importance of establishing certain conditions for promoting mathematical reasoning. These include the role of basic ideas, complex tasks, strands of problems, students’ ownership of their mathematical activity, justification of ideas and student collaborative work. These results suggest potentially interesting insights on problem solving and concept formation. For example, the findings show that attending to basic concepts can challenge students to come up with interesting forms of mathematical reasoning. In this study, the students addressed the challenge to determine the sample space for the World Series Problem by mapping the World Series Problem to the Tower Problem, which they had successfully done in previous years.

The study also shows that providing students with the opportunity to work on the complex task, as opposed to a series of simple tasks taken from the complex task, is crucial for stimulating their mathematical reasoning and the building of durable mathematical knowledge. The idea is that decomposing complex systems into simple subsystems and analyzing them may be a useful way to understand large complex systems. However, it also requires examining relationships among the subsystems (Schoenfeld, 1992). The opportunity to attend to the intricacies of a complex task provides the students with the opportunity to work on unveiling complex mathematical relationships, which enhances deep mathematical understanding.

This study proposes an epistemological distinction between justification and proofs. Justification refers to how students explain their mathematical actions and decisions. Proof is the formal and rigorous argument, which helps mathematicians explain their ideas. The present study highlights the importance of emphasizing justification over rigorous proofs as a way to promote students’ mathematical reasoning. This is consistent with the claim that explanatory proofs need to be emphasized to enhance the building of meaningful mathematical understanding (Hanna, 1990).

The use of strands of problems suggest that work over a series of interrelated problems, as opposed to work on the same problem over and over again promotes mathematical closure. From the perspective of concept formation, the opportunity to revisit the same concepts in different, but related problem situations, helps students build rich and durable forms of mathematical understandings of mathematical concepts. It also provides a way of enhancing reasoning without the need to tell or show students what to do (Maher & Martino, 1996a; Lobato, Clarke, & Ellis, 2005).

The study also shows that the development of a sense of ownership of the mathematical activity enhances the building of personally meaningful mathematical understanding and students’ confidence in their abilities. Similarly, collaborative work among the students in the longitudinal study, shows that students do take it upon themselves to question each other’s ideas and assumptions, which helps the students become flexible in problem comprehension and adaptive rather routine experts at solving problems (Greer, 1997). The students also associate the building of proofs with collaborating, further reinforcing an understanding of mathematical activity as socially discursive.

Finally, it is particularly significant that the students’ reflections on their mathematical activity are consistent with their reflections on their mathematical behaviors in the longitudinal study. The study used a phenomenological interviewing approach, which invites students to choose what aspects of their experiences they consider as relevant and how they want to articulate them. This suggests a parallel development of students’ mathematical behavior and their views about mathematics and learning.

References


Hanna, G. (1990). Explanatory proofs as a way to promote students’ mathematical reasoning. This is consistent with the claim that explanatory proofs need to be emphasized to enhance the building of meaningful mathematical understanding (Hanna, 1990).


